Supplementary Materials

1. Asymptotics in the Simple Example

Figure 5 compares the asymptotic distribution of the LSEs in (2) and (7). First concentrate on the simple model (2). In Example 1, we consider only case (i) with $L(\cdot) = 1$, $\lambda = 1$ and different α values.[9] In the classical case $\alpha = 1$,

$$n\left(\widehat{\gamma}_{LLSE} - \gamma_0\right) \xrightarrow{d} -\text{Exp}\left(1\right) \text{ and } n\left(\widehat{\gamma}_{MLSE} - \gamma_0\right) \xrightarrow{d} \text{DExp}(0, 1/2),$$

where Exp(Δ) is an exponential distribution with scale Δ, and DExp(0,Δ) is a double exponential distribution with location 0 and scale Δ. These are exactly the asymptotic distributions of $\widehat{\gamma}_{LLSE}$ and $\widehat{\gamma}_{MLSE}$ in Section 2 of Yu (2012). When $\alpha = 0.5$, the asymptotic densities of $\widehat{\gamma}_{LLSE}$ and $\widehat{\gamma}_{MLSE}$ at 0 are infinity, while when $\alpha = 2$, the asymptotic density of $\widehat{\gamma}_{LLSE}$ at 0 is zero. This is very different from the classical case where the asymptotic densities of the LSEs are finite and the mode of the asymptotic density of $\widehat{\gamma}_{LLSE}$ is zero. In Example 2,

$$\ln n\left(\widehat{\gamma}_{LLSE} - \gamma_0\right) \xrightarrow{d} -1 \text{ and } \ln n\left(\widehat{\gamma}_{MLSE} - \gamma_0\right) \xrightarrow{d} 0.$$

The asymptotic distributions of the LLSE and MLSE are discrete, and the convergence speed to the corresponding asymptotic distributions is very slow. Here, the asymptotic distribution of $\widehat{\gamma}_{MLSE}$ degenerates. A natural question is whether it has a nondegenerate distribution by increasing the convergence rate.[10] From Gnedenko (1943), it can be shown that

$$n\left(F\left(0\right) - F\left(\frac{t}{\ln^2 n} - \frac{1}{1+\ln n}\right)\right) \to \frac{e^{-(1+t)}}{2}$$

and

$$n\left(F\left(\frac{t}{\ln^2 n} + \frac{1}{1+\ln n}\right) - F\left(0\right)\right) \to \frac{e^{t-1}}{2},$$

so

$$P\left(\ln^2 n\left(\widehat{\gamma}_{LLSE} - \gamma_0 + \frac{1}{1+\ln n}\right) \le t\right) \to e^{-e^{-(1+t)}/2},$$

and

$$P\left(\ln^2 n\left(\widehat{\gamma}_{MLSE} - \gamma_0\right) \le t\right) \to -\int_{-\infty}^{\infty} e^{-e^{-(1+2t-s)}/2} de^{-e^{s-1}/2}.$$

Now, the asymptotic distributions of $\widehat{\gamma}_{LLSE}$ and $\widehat{\gamma}_{MLSE}$ are both continuous. Figure 6 shows the asymptotic distributions of $\widehat{\gamma}_{LLSE}$ and $\widehat{\gamma}_{MLSE}$ with and without the drift normalizing term. Note that although $\ln n\left(\widehat{\gamma}_{MLSE} - \gamma_0\right)$ is degenerate, $\ln^2 n\left(\widehat{\gamma}_{MLSE} - \gamma_0\right)$ is not. When an error term is added in, the convergence rates of both $\widehat{\gamma}_{MLSE}$ and $\widehat{\gamma}_{LLSE}$ are $\ln n$, so adding an error term decreases the convergence rate of $\widehat{\gamma}_{MLSE}$ but not $\widehat{\gamma}_{LLSE}$. The arguments in the case without error term cannot be easily extended to the general case to get a continuous asymptotic distribution in Theorem 4. This is because γ_0 is a "middle" boundary of q. If γ_0 is the conventional one-sided boundary, Theorem 2 of Knight (2001) shows that the estimator can be recentered to get a continuous asymptotic distribution.

When an error term is added in, three common features are shared by all the examples. First, the

[9] In case (ii), the convergence rate changes to $n \ln n$, but the asymptotic distributions remain the same.

[10] Since when the error term is added in, the convergence rates of $\widehat{\gamma}_{LLSE}$ and $\widehat{\gamma}_{MLSE}$ are the same and the asymptotic distribution of $\widehat{\gamma}_{MLSE}$ will not degenerate as long as that of $\widehat{\gamma}_{LLSE}$ does not; our discussion here is only for completeness purpose.

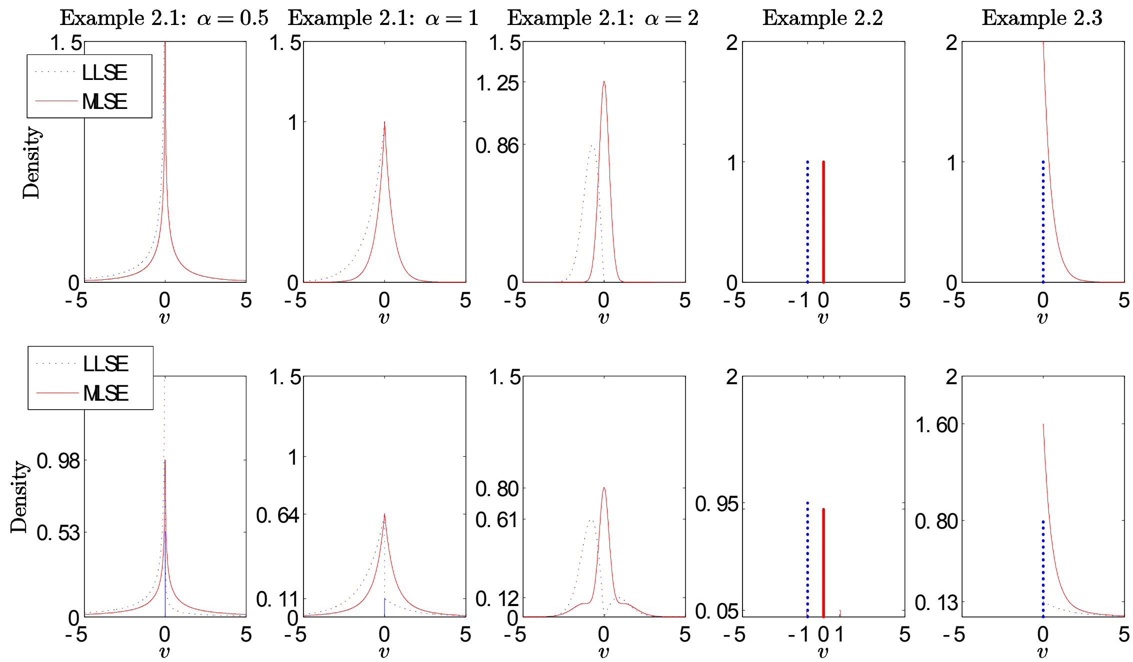

Figure 5: Comparison of the Asymptotic Distribution of the LSEs With and Without the Error Term: The Upper Row for the No-Error-Term Case, the Lower Row for the With-Error-Term Case

asymptotic distributions are more spreading than the case without the error term. Second, the LLSE and the MLSE have the same convergence rate. Third, the support of the asymptotic distribution of the LLSE can include part of the positive axis.

2. Algorithms

This part of supplementary materials develops algorithms to derive the asymptotic distributions of the LSEs of the threshold point. These algorithms extend those in Appendix D of Yu (2012) where $\lambda_1(t) = \lambda_2(t) = f(\gamma_0) \in (0, \infty)$.[11] When simulation methods are used to get the asymptotic distributions, we need to simulate a nonhomogeneous Poisson process. A thinning method is suggested by Lewis and Shedler (1979) for this purpose.

Case (i)

The LLSE

For $t \leq 0$,

$$P(Z_L \leq t) = \sum_{k=0}^{\infty} P(Z_L \leq t \,|MinL = k)\, P(MinL = k) = \sum_{k=0}^{\infty} p_{1k} \cdot P(N_1(|t|) \leq k) \tag{13}$$

[11] Note that even when $0 < \underline{f} \leq f(q) \leq \overline{f} < \infty$ for q in a neighborhood of γ_0, the algorithm for case (i) extends that in Appendix D of Yu (2012) since $\lambda_1(t) = f(\gamma_0-)$ may not equal $\lambda_2(t) = f(\gamma_0+)$.

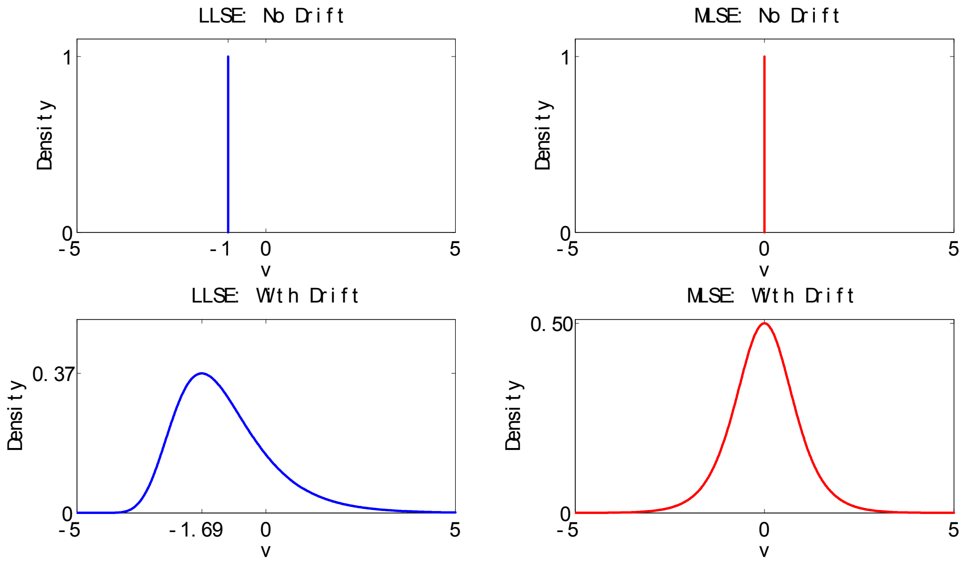

Figure 6: Asymptotic Distributions of $\widehat{\gamma}_{LLSE}$ and $\widehat{\gamma}_{MLSE}$ With and Without the Drift Normalizing Term

where $MinL$ is the number of jumps before attaining the minimum of $D(v)$ on $v \leq 0$, and $p_{1k} = P\left(MinL = k\right)$, $k = 0, 1, \cdots$. For $t > 0$,

$$P\left(0 < Z_L \leq t\right) = \sum_{k=1}^{\infty} P\left(0 < Z_L \leq t \,| MinR = k\,\right) P\left(MinR = k\right) = \sum_{k=1}^{\infty} p_{2k} \cdot P\left(N_2(t) \geq k\right)$$

where $MinR$ is the number of jumps before attaining the minimum of $D(v)$ on $v > 0$, and $p_{2k} = P\left(MinR = k\right)$, $k = 1, 2, \cdots$.

From Appendix D of Yu (2012), $\{p_1; p_2\} \equiv \{p_{10}, p_{11}, \cdots; p_{21}, p_{22}, \cdots\}$ does not depend on $N_1(\cdot)$ and $N_2(\cdot)$ but only on $\phi_1(\cdot)$ and $\phi_2(\cdot)$, where $\phi_\ell(\cdot)$ is the density function of $z_{\ell i}$, $\ell = 1, 2$. So the formulas for $\{p_1; p_2\}$ are similar as those in Yu (2012) and omitted here. The difference lies in the formulas for $P\left(N_1\left(|t|\right) \leq k\right)$ and $P\left(N_2(t) \geq k\right)$:

$$P\left(N_1(-t) \leq k\right) = \sum_{j=0}^{k} \frac{e^{-\Lambda_1(t)} \Lambda_1\left(t\right)^j}{j!}, \text{ and } P\left(N_2(t) \geq k\right) = \sum_{j=k}^{\infty} \frac{e^{-\Lambda_2(t)} \Lambda_2\left(t\right)^j}{j!}.$$

In summary, the cdf of Z_L is

$$F_{Z_L}(t) = \begin{cases} \sum\limits_{k=0}^{\infty} p_{1k} \sum\limits_{j=0}^{k} \frac{e^{-\Lambda_1(t)} \Lambda_1(t)^j}{j!}, & \text{if } t \leq 0; \\ \sum\limits_{k=0}^{\infty} p_{1k} + \sum\limits_{k=1}^{\infty} p_{2k} \sum\limits_{j=k}^{\infty} \frac{e^{-\Lambda_2(t)} \Lambda_2(t)^j}{j!}, & \text{if } t > 0; \end{cases}$$

and the pdf of Z_L is

$$f_{Z_L}(t) = \begin{cases} \lambda_1(t) \sum_{k=0}^{\infty} \frac{e^{-\Lambda_1(t)}\Lambda_1(t)^k}{k!} p_{1,k} = \lambda_1(t) \cdot (\text{Poisson}(\Lambda_1(t)) \circ p_1), & \text{if } t \le 0; \\ \lambda_2(t) \sum_{k=0}^{\infty} \frac{e^{-\Lambda_2(t)}\Lambda_2(t)^k}{k!} p_{2,k+1} = \lambda_2(t) \cdot (\text{Poisson}(\Lambda_2(t)) \circ p_2), & \text{if } t > 0. \end{cases}$$

where $\circ$ means the inner product of two vectors in $\mathbb{R}^\infty$. Z_L is continuously distributed and $\frac{f_{Z_L}(0)}{\lambda_1(0)} = p_{10} \in (0,1)$. When $p_{10} = 1$, $F_{Z_L}(\cdot)$ reduces to (3) in Section 2.

The MLSE

The analysis in (13) is still applicable, but $P(Z_M \le t | MinL = k)$ is different from the LLSE since the middle point of the minimizing interval is taken as the estimator.

First, we derive the distribution of the time length T_{Lk} before the kth jump for $N_1(\cdot)$ and T_{Rk} for $N_2(\cdot)$, $k = 1, 2, \cdots$. For $t > 0$,

$$F_{Lk}(t) \equiv P(T_{Lk} \le t) = P(N_1(t) \ge k) = \sum_{j=k}^{\infty} \frac{e^{-\Lambda_1(-t)}\Lambda_1(-t)^j}{j!},$$

and the density of T_{Lk} is $f_{Lk}(t) = \frac{e^{-\Lambda_1(-t)}\Lambda_1(-t)^{k-1}\lambda_1(-t)}{(k-1)!}$. Similarly, T_{Rk} has the cdf $F_{Rk}(t) = \sum_{j=k}^{\infty} \frac{e^{-\Lambda_2(t)}\Lambda_2(t)^j}{j!}$ and the density $f_{Rk}(t) = \frac{e^{-\Lambda_2(t)}\Lambda_2(t)^{k-1}\lambda_2(t)}{(k-1)!}$. Note that $T_{Lk,k+1} \equiv T_{L,k+1} - T_{Lk}$ is independent of T_{Lk}, $k = 1, 2, \cdots$, so

$$\begin{aligned} & F_{Lk,k+1}(t) \equiv P(T_{L,k+1} - T_{Lk} \le t) \\ = & \int_0^\infty P(T_{L,k+1} - T_{Lk} \le t | T_{Lk}) \, dF_{Lk}(T_{Lk}) \\ = & \int_0^\infty F_{L,k+1}(t + T_{Lk}) \, dF_{Lk}(T_{Lk}) \\ = & \sum_{j=k+1}^{\infty} \int_0^\infty \frac{e^{-\Lambda_1(-t-T_{Lk})}\Lambda_1(-t-T_{Lk})^j}{j!} \frac{e^{-\Lambda_1(-T_{Lk})}\Lambda_1(-T_{Lk})^{k-1}\lambda_1(-T_{Lk})}{(k-1)!} dT_{Lk}, \end{aligned}$$

where the last equality uses Fubini's theorem. Its density

$$f_{Lk,k+1}(t) = \int_0^\infty \frac{e^{-\Lambda_1(-t-T_{Lk})}\Lambda_1(-t-T_{Lk})^k \lambda_1(-t-T_{Lk})}{k!} \frac{e^{-\Lambda_1(-T_{Lk})}\Lambda_1(-T_{Lk})^{k-1} \lambda_1(-T_{Lk})}{(k-1)!} dT_{Lk}.$$

The cdf $F_{Rk,k+1}(\cdot)$ and pdf $f_{Rk,k+1}(\cdot)$ of $T_{Rk,k+1} \equiv T_{R,k+1} - T_{Rk}$ can be similarly derived with $\Lambda_1(\cdot)$ and $\lambda_1(\cdot)$ replaced by $\Lambda_2(\cdot)$ and $\lambda_2(\cdot)$.

For $t \in \mathbb{R}$ and $k = 0$,

$$\begin{aligned} & P(Z_M \le t | MinL = 0) = P\left(\frac{-T_{L1} + T_{R1}}{2} \le t\right) \\ = & \int_{(2t)\vee 0}^{\infty} e^{-\Lambda_1(2t - T_{R1})} dF_{R1}(T_{R1}) = \int_{(2t)\vee 0}^{\infty} e^{-\Lambda_1(2t-T_{R1})} e^{-\Lambda_2(T_{R1})} \lambda_2(T_{R1}) \, dT_{R1}. \end{aligned}$$

For $t \le 0$ and $k > 0$,

$$\begin{aligned} & P\left(Z_M \le t \,|MinL = k\,\right) = P\left(-\tfrac{T_{Lk}+T_{L,k+1}}{2} \le t\right) \\ &= P\left(T_{Lk} + \tfrac{T_{L,k+1}-T_{Lk}}{2} \ge -t\right) \\ &= \textstyle\int_0^{\infty} P\left(T_{Lk} \ge -t - \tfrac{T_{Lk,k+1}}{2} \middle| T_{Lk,k+1}\right) dF_{Lk,k+1}\left(T_{Lk,k+1}\right) \\ &= \textstyle\int_0^{-2t} \left(1 - F_{Lk}\left(-t - \tfrac{T_{Lk,k+1}}{2}\right)\right) dF_{Lk,k+1}\left(T_{Lk,k+1}\right) + 1 - F_{Lk,k+1}\left(-2t\right) \\ &= 1 - \textstyle\int_0^{-2t} F_{Lk}\left(-t - \tfrac{T_{Lk,k+1}}{2}\right) dF_{Lk,k+1}\left(T_{Lk,k+1}\right). \end{aligned} \tag{14}$$

For $t > 0$ and $k > 0$,

$$\begin{aligned} & P\left(0 < Z_M \le t \,|MinR = k\,\right) = P\left(\tfrac{T_{Rk}+T_{R,k+1}}{2} \le t\right) \\ &= P\left(T_{Rk} + \tfrac{T_{Rk,k+1}}{2} \le t\right) = \textstyle\int_0^{2t} F_{Rk}\left(t - \tfrac{T_{Rk,k+1}}{2}\right) dF_{Rk,k+1}\left(T_{Rk,k+1}\right). \end{aligned} \tag{15}$$

In summary, the cdf of Z_M is

$$F_{Z_M}(t) = \begin{cases} p_{10} P\left(Z_M \le t \,|MinL = 0\,\right) + \sum\limits_{k=1}^{\infty} p_{1k} P\left(Z_M \le t \,|MinL = k\,\right), & \text{if } t \le 0; \\ \sum\limits_{k=1}^{\infty} p_{1k} + p_{10} P\left(Z_M \le t \,|MinL = 0\,\right) + \sum\limits_{k=1}^{\infty} p_{2k} P\left(0 < Z_M \le t \,|MinR = k\,\right), & \text{if } t > 0; \end{cases}$$

and the pdf of Z_M is

$$f_{Z_M}(t) = \begin{cases} 2p_{10} \int_0^{\infty} e^{-\Lambda_1(2t - T_{R1})} \lambda_1\left(2t - T_{R1}\right) e^{-\Lambda_2(T_{R1})} \lambda_2\left(T_{R1}\right) dT_{R1} \\ + \sum\limits_{k=1}^{\infty} p_{1k} \int_0^{-2t} f_{Lk}\left(-t - \tfrac{T_{Lk,k+1}}{2}\right) dF_{Lk,k+1}\left(T_{Lk,k+1}\right), \text{ if } t \le 0; \\ 2p_{10} \int_0^{\infty} e^{-\Lambda_1(2t - T_{R1})} \lambda_1\left(2t - T_{R1}\right) e^{-\Lambda_2(T_{R1})} \lambda_2\left(T_{R1}\right) dT_{R1} \\ + \sum\limits_{k=1}^{\infty} p_{2k} \int_0^{2t} f_{Rk}\left(t - \tfrac{T_{Rk,k+1}}{2}\right) dF_{Rk,k+1}\left(T_{Rk,k+1}\right), \quad \text{if } t > 0. \end{cases}$$

When $p_{10} = 1$, $F_{Z_M}(\cdot)$ reduces to (4) in Section 2.

Case (ii)

The support of Z_L and Z_M is $(-\infty, 0]$.

The LLSE

For $t \le 0$,

$$P\left(Z_L \le t\right) = \sum_{k=0}^{\infty} P\left(Z_L \le t \,|MinL = k\,\right) P\left(MinL = k\right) = \sum_{k=0}^{\infty} p'_{1k} \cdot P\left(N_1\left(|t|\right) \le k\right).$$

Note that p'_{1k} is different from p_{1k} since $\sum_{k=0}^{\infty} p'_{1k} = 1 > \sum_{k=0}^{\infty} p_{1k}$. The event $E^{(k)} \equiv \{MinL = k\}$ is equivalent to $\left\{ \sum_{i=1}^{k} z_{1i} \le \sum_{i=1}^{j} z_{1i} \text{ for } j \in \mathbb{Z}_+ \right\}$, where $\sum_{i=1}^{0} \cdot \equiv 0$. It is the intersection of two events:

$$E_1^{(k)} = \left\{ \sum_{i=j}^{k} z_{1i} \le 0,\ j = 1, \cdots, k \right\},\ \text{and } E_2^{(k)} = \left\{ \sum_{i=k+1}^{j} z_{1i} \ge 0,\ j = k+1, \cdots \right\},$$

and these two are independent, so $p'_{1k} = P\left(E_1^{(k)}\right) P\left(E_2^{(k)}\right)$.

Define $E_1^{(k)}(x) = \left\{ \sum_{i=j}^{k} z_{1i} \le x, j = 1, \cdots, k \right\}$. For $x \ge 0$, $P\left(E_1^{(0)}(x)\right) = 1$. When $k \ge 1$,

$$\begin{aligned}
& P\left(E_1^{(k)}(x)\right) \equiv P\left(\sum_{i=j}^{k} z_{1i} \le x,\ j = 1, \cdots, k \right) \\
= & \int_{-\infty}^{x} \int_{-\infty}^{x - z_{1k}} \cdots \int_{-\infty}^{x - \sum_{j=3}^{k} z_{1j}} \left[\int_{-\infty}^{x - \sum_{j=2}^{k} z_{1j}} \phi_1(z_{11})\, dz_{11} \right] \phi_1(z_{12}) \cdots \phi_1(z_{1,k-1})\, \phi_1(z_{1,k})\, dz_{12} \cdots dz_{1k-1} dz_{1k} \\
= & \int_{-\infty}^{x} P\left(E_1^{(k-1)}(x - z_{1,k})\right) \phi_1(z_{1,k})\, dz_{1k} = \int_{0}^{\infty} P\left(E_1^{(k-1)}(t)\right) \phi_1(x - t)\, dt,
\end{aligned}$$

which is a recursive solution.

Define $P\left(E_2^{(k)}\right) = 1 - F_1(0)$, then

$$\begin{aligned}
& 1 - F_1(x) \equiv P\left(\sum_{i=k+1}^{j} z_{1i} \ge x, j = k+1, \cdots \right) \\
& = \int_x^{\infty} \phi_1(z_{1,k+1})\, P\left(\sum_{i=k+2}^{j} z_{1i} \ge x - z_{1,k+1}, j = k+2, \cdots \right) dz_{1,k+1} \\
& = \int_x^{\infty} \phi_1(z_{1,k+1}) \left(1 - F_1(x - z_{1,k+1})\right) dz_{1,k+1} \\
& = \int_{-\infty}^{0} \phi_1(x - t) \left(1 - F_1(t)\right) dt.
\end{aligned} \tag{16}$$

This is an integral equation called the homogeneous Wiener-Hopf equation of the second kind with boundary condition $1 - F_1(-\infty) = 1$, $1 - F_1(\infty) = 0$ since $F_1(\cdot)$ is the cdf of $\min\left\{ \sum_{i=k+1}^{j} z_{1i}, j = k+1, \cdots \right\}$ and does not depend on k.

In summary, the cdf of Z_L is

$$F_{Z_L}(t) = \begin{cases} \sum_{k=0}^{\infty} p'_{1k} \sum_{j=0}^{k} \frac{e^{-\Lambda_1(t)} \Lambda_1(t)^j}{j!}, & \text{if } t \le 0; \\ 1, & \text{if } t > 0; \end{cases}$$

and the pdf of Z_L is

$$f_{Z_L}(t) = \begin{cases} \lambda_1(t) \sum_{k=0}^{\infty} \frac{e^{-\Lambda_1(t)} \Lambda_1(t)^k}{k!} p'_{1,k} = \lambda_1(t) \cdot \left(\text{Poisson}(\Lambda_1(t)) \circ p'_1\right), & \text{if } t \le 0; \\ 0, & \text{if } t > 0; \end{cases}$$

where $p_1' = \{p_{10}', p_{11}', \cdots\}$. Z_L is continuously distributed and $\frac{f_{Z_L}(0)}{\lambda_1(0)} = p_{10}' = 1 - F_1(0)$.

The MLSE

The derivation is similar to Case (i). For $t \le 0$ and $k = 0$,

$$P\left(Z_M \le t \,|MinL = 0\right) = P\left(\frac{-T_{L1}}{2} \le t\right) = P\left(T_{L1} \ge -2t\right) = e^{-\Lambda_1(-2t)}.$$

For $k > 0$, $P\left(Z_M \le t\,|MinL = k\right)$ is the same as (14). In summary, the cdf of Z_M is

$$F_{Z_M}(t) = \begin{cases} p_{10}' e^{-\Lambda_1(-2t)} + \sum\limits_{k=1}^{\infty} p_{1k}' P\left(Z_M \le t\,|MinL = k\right), & \text{if } t \le 0; \\ 1, & \text{if } t > 0; \end{cases}$$

and the pdf of Z_M is

$$f_{Z_M}(t) = \begin{cases} -2p_{10}' e^{-\Lambda_1(-2t)} \lambda_1\left(-2t\right) + \sum\limits_{k=1}^{\infty} p_{1k}' \int_0^{-2t} f_{Lk}\left(-t - \frac{T_{Lk,k+1}}{2}\right) dF_{Lk,k+1}\left(T_{Lk,k+1}\right), & \text{if } t \le 0; \\ 0, & \text{if } t > 0. \end{cases}$$

Case (iii)

The support of Z_L and Z_M is $[0, \infty)$.

The LLSE

First, there is a point mass at zero in the distribution of Z_L:

$$P\left(Z_L = 0\right) = P\left(MinR = 0\right) = 1 - F_2(0) \equiv p_{20}',$$

where $F_2(\cdot)$ is the cdf of $\min\left\{\sum\limits_{i=1}^{j} z_{2i}, j = 1, 2, \cdots\right\}$. For $t > 0$,

$$P\left(0 < Z_L \le t\right) = \sum_{k=1}^{\infty} P\left(0 < Z_L \le t\,|MinR = k\right) P\left(MinR = k\right) = \sum_{k=1}^{\infty} p_{2k}' \cdot P\left(N_2(t) \ge k\right),$$

where p_{2k}' is different from p_{2k} and can be calculated in a similar way as in Case (ii). So the cdf of Z_L is a mixture of continuous and discrete:

$$F_{Z_L}(t) = \begin{cases} 0, & \text{if } t < 0; \\ p_{20}', & \text{if } t = 0; \\ p_{20}' + \sum\limits_{k=1}^{\infty} p_{2k}' \sum\limits_{j=k}^{\infty} \frac{e^{-\Lambda_2(t)} \Lambda_2(t)^j}{j!}, & \text{if } t > 0; \end{cases}$$

and the Randon-Nikodym derivative of F_{Z_L} with respect to the Lebesgue measure plus a counting measure at zero is

$$f_{Z_L}(t) = \begin{cases} 0, & \text{if } t < 0; \\ p_{20}', & \text{if } t = 0; \\ \lambda_2\left(t\right) \sum\limits_{k=0}^{\infty} \frac{e^{-\Lambda_2(t)} \Lambda_2(t)^k}{k!} p_{2,k+1}' = \lambda_2\left(t\right) \cdot \left(\text{Poisson}(\Lambda_2\left(t\right)) \circ p_2'\right), & \text{if } t > 0; \end{cases}$$

where $p_2' = \{p_{21}', p_{22}', \cdots\}$.

The MLSE

The derivation is similar to Case (ii). For $t > 0$ and $k = 0$,

$$P\left(0 < Z_M \le t\,|MinR = 0\right) = P\left(\frac{T_{R1}}{2} \le t\right) = \sum_{j=1}^{\infty} \frac{e^{-\Lambda_2(2t)}\Lambda_2\left(2t\right)^j}{j!}.$$

For $k > 0$, $P\left(Z_M \le t\,|MinR = k\right)$ is the same as (15). In summary, the cdf of Z_M is

$$F_{Z_M}(t) = \begin{cases} 0, & \text{if } t \le 0; \\ p_{20}' \sum\limits_{j=1}^{\infty} \frac{e^{-\Lambda_2(2t)}\Lambda_2(2t)^j}{j!} + \sum\limits_{k=1}^{\infty} p_{2k}' P\left(Z_M \le t\,|MinR = k\right), & \text{if } t > 0; \end{cases}$$

and the pdf of Z_M is

$$f_{Z_M}(t) = \begin{cases} 0, & \text{if } t \le 0; \\ 2p_{20}' e^{-\Lambda_2(2t)}\lambda_2\left(2t\right) + \sum\limits_{k=1}^{\infty} p_{2k}' \int_0^{2t} f_{Rk}\left(t - \frac{T_{Rk,k+1}}{2}\right) dF_{Rk,k+1}\left(T_{Rk,k+1}\right), & \text{if } t > 0. \end{cases}$$

Case (iv)

First, the distributions of $D(-1)$ and $D(1)$ are derived. For $t < 0$,

$$\begin{aligned} & P\left(\sum\nolimits_{i=1}^{N_\ell(\kappa_\ell)} z_{\ell i} \le t\right) = \sum_{k=0}^{\infty} P\left(\sum\nolimits_{i=1}^{k} z_{\ell i} \le t | N_\ell\left(\kappa_\ell\right) = k\right) P\left(N_\ell\left(\kappa_\ell\right) = k\right) \\ = & \sum_{k=1}^{\infty} \frac{e^{-\kappa_\ell}\kappa_\ell^k}{k!} P\left(\sum\nolimits_{i=1}^{k} z_{\ell i} \le t\right) = \sum_{k=1}^{\infty} \frac{e^{-\kappa_\ell}\kappa_\ell^k}{k!} \Phi_\ell^k\left(t\right), \end{aligned}$$

and for $t \ge 0$,

$$P\left(\sum\nolimits_{i=1}^{N_\ell(\kappa_\ell)} z_{\ell i} \le t\right) = e^{-\kappa_\ell} + \sum_{k=1}^{\infty} \frac{e^{-\kappa_\ell}\kappa_\ell^k}{k!} \Phi_\ell^k\left(t\right)$$

where $\Phi_\ell^k\left(\cdot\right)$ is the kth-order convolution of the cdf of z_ℓ. So there is a point mass $e^{-\kappa_\ell}$ at zero in the distribution of $\sum\nolimits_{i=1}^{N_\ell(\kappa_\ell)} z_{\ell i}$. Cases (v) and (vi) are omitted since they can be derived easily from the cdf of $\sum\nolimits_{i=1}^{N_\ell(\kappa_\ell)} z_{\ell i}$.

The LLSE

$$\begin{aligned} & P\left(c_n\left(\widehat{\gamma}_{LLSE} - \gamma_0\right) = -1\right) \to P\left(D(1) \ge \min\left\{D(-1), 0\right\}\right) \\ = & P\left(D(1) \ge D(-1), D(-1) < 0\right) + P\left(D(1) \ge 0, D(-1) \ge 0\right) \\ = & \int_{-\infty}^{0} P\left(D(1) \ge s\right)\big|_{s=D(-1)} dP\left(D(-1) \le t\right) + P\left(D(1) \ge 0\right) P\left(D(-1) \ge 0\right) \\ = & \int_{-\infty}^{0} \left[e^{-\kappa_2} + \sum_{k=1}^{\infty} \frac{e^{-\kappa_2}\kappa_2^k}{k!}\left(1 - \Phi_2^k\left(t\right)\right)\right] \left[\sum_{j=1}^{\infty} \frac{e^{-\kappa_1}\kappa_1^j}{j!} \phi_1^j\left(t\right)\right] dt \end{aligned}$$

$$+\left[e^{-\kappa_2}+\sum_{k=1}^{\infty}\frac{e^{-\kappa_2}\kappa_2^k}{k!}\left(1-\Phi_2^k(0)\right)\right]\left[e^{-\kappa_1}+\sum_{j=1}^{\infty}\frac{e^{-\kappa_1}\kappa_1^j}{j!}\left(1-\Phi_1^j(0)\right)\right],$$

and

$$\begin{aligned}
& P\left(c_n\left(\widehat{\gamma}_{LLSE}-\gamma_0\right)=1\right)\rightarrow P\left(D(1)<\min\left\{D(-1),0\right\}\right)\\
= & P\left(D(1)<D(-1),D(-1)<0\right)+P\left(D(1)<0,D(-1)\geq 0\right)\\
= & \int_{-\infty}^{0}\left[\sum_{k=1}^{\infty}\frac{e^{-\kappa_2}\kappa_2^k}{k!}\Phi_2^k(t)\right]\left[\sum_{j=1}^{\infty}\frac{e^{-\kappa_1}\kappa_1^j}{j!}\phi_1^j(t)\right]dt\\
& +\left(\sum_{k=1}^{\infty}\frac{e^{-\kappa_2}\kappa_2^k}{k!}\Phi_2^k(0)\right)\left[e^{-\kappa_1}+\sum_{j=1}^{\infty}\frac{e^{-\kappa_1}\kappa_1^j}{j!}\left(1-\Phi_1^j(0)\right)\right],
\end{aligned}$$

where $\phi_\ell^k(\cdot)$ is the kth-order convolution of the pdf of z_ℓ.

The MLSE

$$\begin{aligned}
& P\left(c_n\left(\widehat{\gamma}_{MLSE}-\gamma_0\right)=-1\right)\rightarrow P\left(D(1)>D(-1),D(-1)<0\right)\\
= & \int_{-\infty}^{0}\left[e^{-\kappa_2}+\sum_{k=1}^{\infty}\frac{e^{-\kappa_2}\kappa_2^k}{k!}\left(1-\Phi_2^k(t)\right)\right]\left[\sum_{j=1}^{\infty}\frac{e^{-\kappa_1}\kappa_1^j}{j!}\phi_1^j(t)\right]dt,
\end{aligned}$$

and

$$\begin{aligned}
& P\left(c_n\left(\widehat{\gamma}_{MLSE}-\gamma_0\right)=0\right)\rightarrow P\left(D(1)\geq 0,D(-1)\geq 0\right),\\
= & \left[e^{-\kappa_2}+\sum_{k=1}^{\infty}\frac{e^{-\kappa_2}\kappa_2^k}{k!}\left(1-\Phi_2^k(0)\right)\right]\left[e^{-\kappa_1}+\sum_{j=1}^{\infty}\frac{e^{-\kappa_1}\kappa_1^j}{j!}\left(1-\Phi_1^j(0)\right)\right].
\end{aligned}$$

The limit of $P\left(c_n\left(\widehat{\gamma}_{MLSE}-\gamma_0\right)=1\right)$ is the same as that of $P\left(c_n\left(\widehat{\gamma}_{LLSE}-\gamma_0\right)=1\right)$.

3. Extra Simulation Results

We report extra simulation results beyond those in Section 4.2 in this section. Table 3 and 4 report the simulation result for $\beta_0=0.5$ and 2 in model (7), respectively. Table 5 includes the MAE of all estimators in model (8).

Additional References

Knight, K., 2001, Limiting Distributions of Linear Programming Estimators, *Extremes*, 4, 87-103.

Lewis, P.A.W. and G.S. Shedler, 1979, Simulation of Nonhomogeneous Poisson Processes by Thinning, *Naval Research Logistics Quarterly*, 26, 403-413.

Risk ($\times 10^{-2}$) $\rightarrow$	RMSE				MAE			
Examples$\downarrow$ Estimators$\rightarrow$	$\widehat{\gamma}_{LLSE}$	$\widehat{\gamma}_{MLSE}$	$\widehat{\beta}$	$\widehat{\beta}_o$	$\widehat{\gamma}_{LLSE}$	$\widehat{\gamma}_{MLSE}$	$\widehat{\beta}$	$\widehat{\beta}_o$
$n = 100$								
Example 1: $\alpha = 0.5$	1.583	1.609	7.818	7.229	0.058	0.053	5.198	4.659
Example 1: $\alpha = 1$	5.991	5.968	7.657	7.119	2.002	1.739	5.033	4.871
Example 1: $\alpha = 2$	20.030	19.413	7.682	7.205	15.052	14.124	4.827	4.683
Example 2	54.325	56.517	7.662	6.975	31.378	30.014	4.916	4.749
Example 3	5.887	6.120	7.168	6.936	0	0.817	4.763	4.619
$n = 400$								
Example 1: $\alpha = 0.5$	0.057	0.058	3.711	3.650	0.003	0.002	2.568	2.464
Example 1: $\alpha = 1$	1.480	1.454	3.576	3.546	0.539	0.475	2.427	2.415
Example 1: $\alpha = 2$	9.463	9.014	3.620	3.527	6.803	5.984	2.443	2.367
Example 2 ($n = 1000$)	19.303	17.564	2.311	2.293	18.147	17.393	1.492	1.508
Example 3	1.069	1.139	3.615	3.590	0	0.214	2.481	2.455

Table 3: Performances of $\widehat{\gamma}$ and $\widehat{\beta}$: $\beta_0 = 0.5$ (Based on 1000 Repetitions)

Risk ($\times 10^{-2}$) $\rightarrow$	RMSE				MAE			
Examples$\downarrow$ Estimators$\rightarrow$	$\widehat{\gamma}_{LLSE}$	$\widehat{\gamma}_{MLSE}$	$\widehat{\beta}$	$\widehat{\beta}_o$	$\widehat{\gamma}_{LLSE}$	$\widehat{\gamma}_{MLSE}$	$\widehat{\beta}$	$\widehat{\beta}_o$
$n = 100$								
Example 1: $\alpha = 0.5$	0.056	0.039	7.018	6.989	0.005	0.006	4.816	4.806
Example 1: $\alpha = 1$	1.307	0.769	7.251	7.231	0.677	0.380	4.602	4.662
Example 1: $\alpha = 2$	10.245	4.309	7.208	7.176	8.498	2.315	4.945	4.945
Example 2	24.550	7.676	7.112	7.113	23.081	3.106	4.648	4.698
Example 3	0.182	0.715	7.151	7.157	0	0.358	4.847	4.847
$n = 400$								
Example 1: $\alpha = 0.5$	0.003	0.003	3.518	3.518	0.0003	0.0004	2.351	2.374
Example 1: $\alpha = 1$	0.359	0.201	3.480	3.479	0.183	0.092	2.347	2.344
Example 1: $\alpha = 2$	5.152	1.993	3.602	3.596	4.213	1.090	2.390	2.390
Example 2 ($n = 1000$)	15.540	4.092	2.207	2.207	15.379	1.323	1.482	1.481
Example 3	0.056	0.195	3.531	3.530	0	0.096	2.286	2.293

Table 4: Performances of $\widehat{\gamma}$ and $\widehat{\beta}$: $\beta_0 = 2$ (Based on 1000 Repetitions)

Examples$\downarrow$ Estimators$\rightarrow$	$\widehat{\gamma}_{LLSE}$	$\widehat{\gamma}_{MLSE}$	$SATE$	$SATE_o$	$SATT$	$SATTE_o$
$n = 100$						
Example 1: $\alpha = 0.5$	0.006	0.007	15.383	15.353	11.856	11.856
Example 1: $\alpha = 1$	0.758	0.449	13.901	13.962	12.216	12.155
Example 1: $\alpha = 2$	8.848	2.801	15.522	15.318	12.022	12.020
Example 2	24.188	4.080	15.543	15.666	12.736	12.710
Example 3	0.750	1.304	15.580	15.512	10.167	10.264
$n = 400$						
Example 1: $\alpha = 0.5$	0.0004	0.0005	7.691	7.688	6.121	6.160
Example 1: $\alpha = 1$	0.190	0.116	7.768	7.737	6.196	6.227
Example 1: $\alpha = 2$	4.326	1.391	7.659	7.590	6.361	6.405
Example 2 ($n = 1000$)	15.492	1.610	4.731	4.724	3.666	3.672
Example 3	0.189	0.324	7.336	7.344	4.823	4.752

Table 5: MAE of $\widehat{\gamma}$ and the SATE and SATT in 10^{-2} (Based on 1000 Repetitions)