Appendix: Lemmas to Theorem 1

Throughout the appendix, $\approx$ means higher order terms are neglected, C refers to a positive constant which may not be the same at each occurrence, and $\rightsquigarrow$ signifies the weak convergence of a stochastic process on the relevant probability space.

Lemma 1 *$\widehat{v} = O_p(1)$ as $\|\delta_0\| \to 0$.*

Proof. Take $v \geq 0$ for illustration and $v < 0$ can be similarly proved. Partition $\mathbb{R}_+ := \{v|v \geq 0\}$ into the "shells" $S_j = \left\{v : 2^{j-1} < v \leq 2^j\right\}$ with j ranging over the integers. Given an integer J,

$$P\left(\widehat{v} > 2^J\right) \leq \sum_{j \geq J} P\left(\inf_{v \in S_j}\left(D\left(\frac{v}{\|\delta_0\|^2}\right)\right) \leq D(0) = 0\right).$$

Now,

$$\begin{aligned}
&P\left(\inf_{v \in S_j}\left(D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right) \leq 0\right)\\
\leq\; &P\left(\inf_{v \in S_j}\left(\mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right] - \left|D\left(\tfrac{v}{\|\delta_0\|^2}\right) - \mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right]\right|\right) \leq 0\right)\\
\leq\; &P\left(\sup_{v \in S_j}\left|D\left(\tfrac{v}{\|\delta_0\|^2}\right) - \mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right]\right| \geq \inf_{v \in S_j}\mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right]\right)\\
\leq\; &\mathbb{E}\left[\sup_{v \in S_j}\left|D\left(\tfrac{v}{\|\delta_0\|^2}\right) - \mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right]\right|^2\right] \Big/ \inf_{v \in S_j}\mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right]^2,
\end{aligned}$$

where the first equality is because

$$D\left(\tfrac{v}{\|\delta_0\|^2}\right) - \mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right] \geq -\left|D\left(\tfrac{v}{\|\delta_0\|^2}\right) - \mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right]\right|$$

which implies $\mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right] - \left|D\left(\tfrac{v}{\|\delta_0\|^2}\right) - \mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right]\right| \leq D\left(\tfrac{v}{\|\delta_0\|^2}\right)$, the second equality is because

$$\inf_{v \in S_j}\left(\mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right] - \left|D\left(\tfrac{v}{\|\delta_0\|^2}\right) - \mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right]\right|\right) \geq \inf_{v \in S_j}\mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right] - \sup_{v \in S_j}\left|D\left(\tfrac{v}{\|\delta_0\|^2}\right) - \mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right]\right|,$$

and the last equality is from Markov's inequality. Note that the demeaned process $\widetilde{D}(v) := D\left(\tfrac{v}{\|\delta_0\|^2}\right) - \mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right]$ satisfies for any $s \leq t$,

$$\begin{aligned}
\mathbb{E}\left[\widetilde{D}(t)\Big|\left\{\widetilde{D}(\tau)\right\}_{\tau \leq s}\right] &= \mathbb{E}\left[\widetilde{D}(t) - \widetilde{D}(s)\Big|\left\{\widetilde{D}(\tau)\right\}_{\tau \leq s}\right] + \mathbb{E}\left[\widetilde{D}(s)\Big|\left\{\widetilde{D}(\tau)\right\}_{\tau \leq s}\right]\\
&= \mathbb{E}\left[\widetilde{D}(s)\Big|\left\{\widetilde{D}(\tau)\right\}_{\tau \leq s}\right] = \widetilde{D}(s),
\end{aligned}$$

where the second to last equality is because $\widetilde{D}(v)$ is an independent increments process with mean zero such that $\widetilde{D}(t) - \widetilde{D}(s)$ is independent of $\left\{\widetilde{D}(\tau)\right\}_{\tau \leq s}$ and has mean zero. So $\widetilde{D}(v)$ is a continuous martingale indexed by v and $\left|\widetilde{D}(v)\right|$ is a submartingale. To calculate $\mathbb{E}\left[\sup_{v \in S_j}\left|\widetilde{D}(v)\right|^2\right]$, we apply Doob's martingale inequality (see, e.g., Theorem 20 of Protter (2004)). First, by Assumption (iii),

$$\mathbb{E}[z_{2i}] = \mathbb{E}\left[-2x_i'\delta_0 u_{2i} + \delta_0 x_i x_i'\delta_0 | q_i = \gamma_0+\right] = \delta_0\mathbb{E}[xx|q = \gamma_0]\,\delta_0 =: \|\delta_0\|^2 Q_\delta,$$

and

$$\begin{aligned}
&\mathbb{V}(z_{2i}) = \mathbb{V}(-2x_i'\delta_0 u_{2i} + \delta_0 x_i x_i'\delta_0 | q_i = \gamma_0 +) \\
&= \mathbb{E}\left[\mathbb{V}(-2x_i'\delta_0 u_{2i} + \delta_0 x_i x_i'\delta_0 | x_i, q_i = \gamma_0 +) | q_i = \gamma_0 +\right] + \mathbb{V}\left(\mathbb{E}\left[-2x_i'\delta_0 u_{2i} + \delta_0 x_i x_i'\delta_0 | x_i, q_i = \gamma_0 +\right] | q_i = \gamma_0 +\right) \\
&= \mathbb{E}\left[\left(-2x_i'\delta_0 u_{2i}\right)^2 | q_i = \gamma_0\right] + \mathbb{V}\left(\delta_0 x_i x_i'\delta_0 | q_i = \gamma_0\right) \\
&= 4\delta_0 \mathbb{E}\left[x x u_2^2 | q = \gamma_0\right]\delta_0 + O\left(\|\delta_0\|^4\right) =: 4\|\delta_0\|^2 \Omega_{2\delta},
\end{aligned}$$

where $Q_\delta = \frac{\delta_0 \mathbb{E}[xx|q=\gamma_0]\delta_0}{\delta_0'\delta_0} \to Q$ and $\Omega_{2\delta} = \frac{\delta_0 \mathbb{E}\left[xxu_2^2|q=\gamma_0\right]\delta_0}{\delta_0'\delta_0} + O\left(\|\delta_0\|^2\right) \to \Omega_2$ as $\|\delta_0\| \to 0$ from Assumption (i). Now, by Doob's martingale inequality,

$$\begin{aligned}
\mathbb{E}\left[\sup_{v\in S_j}\left|D\left(\frac{v}{\|\delta_0\|^2}\right) - \mathbb{E}\left[D\left(\frac{v}{\|\delta_0\|^2}\right)\right]\right|^2\right] &\leq 4\mathbb{E}\left[\left|D\left(\frac{2^j}{\|\delta_0\|^2}\right) - \mathbb{E}\left[D\left(\frac{2^j}{\|\delta_0\|^2}\right)\right]\right|^2\right] \\
&= 16 f_q(\gamma_0) 2^j \Omega_2 + o(1),
\end{aligned}$$

where the equality is because

$$\begin{aligned}
&\mathbb{V}\left(D\left(\tfrac{v}{\|\delta_0\|^2}\right)\right) = \mathbb{E}\left[\mathbb{V}\left(D\left(\tfrac{v}{\|\delta_0\|^2}\right)\middle| N_2(\cdot)\right)\right] + \mathbb{V}\left(\mathbb{E}\left[D\left(\tfrac{v}{\|\delta_0\|^2}\right)\middle| N_2(\cdot)\right]\right) \\
&= \mathbb{E}\left[N_2\left(\tfrac{v}{\|\delta_0\|^2}\right)\mathbb{V}(z_{2i})\right] + \mathbb{V}\left(N_2\left(\tfrac{v}{\|\delta_0\|^2}\right)\mathbb{E}[z_{2i}]\right) \\
&= \tfrac{f_q(\gamma_0)v}{\|\delta_0\|^2}\mathbb{V}(z_{2i}) + \tfrac{f_q(\gamma_0)v}{\|\delta_0\|^2}\mathbb{E}[z_{2i}]^2 = \tfrac{f_q(\gamma_0)v}{\|\delta_0\|^2}\mathbb{E}\left[z_{2i}^2\right] \\
&= f_q(\gamma_0)v\left(4\Omega_{2\delta} + \|\delta_0\|^2 Q_\delta^2\right) = 4 f_q(\gamma_0) v\Omega_2 + o(1),
\end{aligned}$$

with the last equality from Assumption (i) and $\|\delta_0\| \to 0$. Since

$$\begin{aligned}
\mathbb{E}\left[D\left(\frac{v}{\|\delta_0\|^2}\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[D\left(\frac{v}{\|\delta_0\|^2}\right)\middle| N_2(\cdot)\right]\right] = \mathbb{E}\left[N_2\left(\frac{v}{\|\delta_0\|^2}\right)\mathbb{E}[z_{2i}]\right] \\
&= f_q(\gamma_0) v \frac{\mathbb{E}[z_{2i}]}{\|\delta_0\|^2} = f_q(\gamma_0) v Q_\delta = f_q(\gamma_0) v Q + o(1),
\end{aligned}$$

we have

$$P\left(\inf_{v\in S_j}\left(D\left(\frac{v}{\|\delta_0\|^2}\right)\right) \leq 0\right) \leq \frac{16 f_q(\gamma_0) 2^j \Omega_2}{\left(2^{j-1} f_q(\gamma_0) Q\right)^2} = \frac{C}{2^{j-1}},$$

where $C = 32\Omega_2 / f_q(\gamma_0) Q^2$ is a positive constant. As a result,

$$P\left(\widehat{v} > 2^J\right) \leq \sum_{j\geq J}\frac{C}{2^{j-1}} \to 0$$

as $J \to \infty$ and the proof is complete. ■

Lemma 2 $D\left(\frac{v}{\|\delta_0\|^2}\right) \rightsquigarrow C(v) \in \mathbf{C}_{\min}(\mathbb{R})$ *as* $\|\delta_0\| \to 0$.

Proof. As in Lemma 1, take $v > 0$. Define

$$Z_\delta(v) = \frac{D\left(\frac{v}{\|\delta_0\|^2}\right) - \mathbb{E}\left[D\left(\frac{v}{\|\delta_0\|^2}\right)\right]}{\sqrt{\mathbb{V}\left(D\left(\frac{v}{\|\delta_0\|^2}\right)\right)\Big/ v}} = \frac{D\left(\frac{v}{\|\delta_0\|^2}\right) - \mu_\delta v}{\sigma_\delta},$$

where $\mu_\delta = f_q(\gamma_0) Q_\delta$ and $\sigma_\delta^2 = f_q(\gamma_0)\left(4\Omega_{2\delta} + \|\delta_0\|^2 Q_\delta^2\right)$. Since $\mu_\delta \to f_q(\gamma_0) Q$ and $\sigma_\delta^2 \to 4 f_q(\gamma_0)\Omega_2$, by Slutsky's theorem, we need only show that $Z_\delta(v) \rightsquigarrow W_2(v)$. We check the two conditions in Theorem 2.3 of Kim and Pollard. Specifically, (i) fidi-convergence: for any $v_2 > v_1 > 0$ and $t_1, t_2 \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left\{\sqrt{-1}\left[t_1 Z_\delta(v_1) + t_2 Z_\delta(v_2)\right]\right\}\right] \longrightarrow \exp\left\{-\frac{1}{2}(t_1, t_2)\begin{pmatrix} v_1 & v_1 \\ v_1 & v_2 \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right\};$$

and (ii) stochastic equicontinuity: for any $\epsilon > 0, \eta > 0$, there exist a $\Delta > 0$ such that

$$\overline{\lim_{n\to\infty}} P\left(\sup_{|v_2 - v_1| < \Delta} |Z_\delta(v_2) - Z_\delta(v_1)| > \epsilon\right) < \eta.$$

To prove (i), note that

$$\begin{aligned}
& \mathbb{E}\left[\exp\left\{\sqrt{-1}\left[t_1 Z_\delta(v_1) + t_2 Z_\delta(v_2)\right]\right\}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\exp\left\{\sqrt{-1}\left[t_1 Z_\delta(v_1) + t_2 Z_\delta(v_2)\right]\right\}\middle| N_2(\cdot)\right]\right] \\
&= \exp\left\{\sqrt{-1}\left[-t_1 \tfrac{\mu_\delta v_1}{\sigma_\delta} - t_2 \tfrac{\mu_\delta v_2}{\sigma_\delta}\right]\right\} \cdot \mathbb{E}\left[\exp\left\{\sqrt{-1}\left[t_1 \frac{\sum_{i=1}^{N_2\left(\frac{v_1}{\|\delta_0\|^2}\right)} z_{2i}}{\sigma_\delta} + t_2 \frac{\sum_{i=1}^{N_2\left(\frac{v_2}{\|\delta_0\|^2}\right)} z_{2i}}{\sigma_\delta}\right]\right\}\right] \\
&= \exp\left\{\sqrt{-1}\left[-t_1 \tfrac{\mu_\delta v_1}{\sigma_\delta} - t_2 \tfrac{\mu_\delta v_2}{\sigma_\delta}\right]\right\} \cdot \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^{N_2\left(\frac{v_1}{\|\delta_0\|^2}\right)} \mathbb{E}\left[\exp\left\{\sqrt{-1}(t_1 + t_2)\tfrac{z_{2i}}{\sigma_\delta}\right\}\right]\middle| N_2(\cdot)\right]\right] \\
&\quad \cdot \mathbb{E}\left[\mathbb{E}\left[\prod_{i=N_2\left(\frac{v_1}{\|\delta_0\|^2}\right)+1}^{N_2\left(\frac{v_2}{\|\delta_0\|^2}\right)} \mathbb{E}\left[\exp\left\{\sqrt{-1} t_2 \tfrac{z_{2i}}{\sigma_\delta}\right\}\right]\middle| N_2(\cdot)\right]\right] \\
&= \exp\left\{\sqrt{-1}\left[-t_1 \tfrac{\mu_\delta v_1}{\sigma_\delta} - t_2 \tfrac{\mu_\delta v_2}{\sigma_\delta}\right]\right\} \cdot \sum_{k=0}^{\infty}\left\{\frac{e^{-f_q(\gamma_0)v_1/\|\delta_0\|^2}\left(f_q(\gamma_0)\frac{v_1}{\|\delta_0\|^2}\right)^k}{k!}\prod_{i=1}^{k}\mathbb{E}\left[\exp\left\{\sqrt{-1}(t_1 + t_2)\tfrac{z_{2i}}{\sigma_\delta}\right\}\right]\right\} \\
&\quad \cdot \sum_{k=0}^{\infty}\left\{\frac{e^{-f_q(\gamma_0)(v_2 - v_1)/\|\delta_0\|^2}\left(f_q(\gamma_0)\frac{v_2 - v_1}{\|\delta_0\|^2}\right)^k}{k!}\prod_{i=1}^{k}\mathbb{E}\left[\exp\left\{\sqrt{-1} t_2 \tfrac{z_{2i}}{\sigma_\delta}\right\}\right]\right\} \\
&=: T_1 \cdot T_2 \cdot T_3,
\end{aligned}$$

where $\prod_{i=1}^{0} \cdot := 1$. Because $\|\delta_0\| \to 0$, we take Taylor expansion of

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{\sqrt{-1}(t_1 + t_2)\frac{z_{2i}}{\sigma_\delta}\right\}\right] &= \mathbb{E}\left[\exp\left\{\sqrt{-1}(t_1 + t_2)\|\delta_0\|\frac{z_{2i}/\|\delta_0\|}{\sigma_\delta}\right\}\right] \\
&=: \mathbb{E}\left[\exp\left\{\sqrt{-1}(t_1 + t_2)\|\delta_0\| Z_{2i}\right\}\right] := g((t_1 + t_2)\|\delta_0\|)
\end{aligned}$$

about $(t_1 + t_2)\|\delta_0\|$ at 0, giving

$$\begin{aligned}
& \mathbb{E}\left[\exp\left\{\sqrt{-1}(t_1 + t_2)\|\delta_0\| Z_{2i}\right\}\right] \\
= \; & 1 + \sqrt{-1}(t_1 + t_2)\|\delta_0\|\mathbb{E}[Z_{2i}] - \frac{1}{2}(t_1 + t_2)^2\|\delta_0\|^2\mathbb{E}\left[Z_{2i}^2\right] + o\left((t_1 + t_2)^2\|\delta_0\|^2\right) \\
= \; & 1 + \sqrt{-1}(t_1 + t_2)\frac{\mathbb{E}[z_{2i}]}{\sigma_\delta} - \frac{1}{2}(t_1 + t_2)^2\frac{\mathbb{E}\left[z_{2i}^2\right]}{\sigma_\delta^2} + o\left(\|\delta_0\|^2\right),
\end{aligned}$$

where $E\left[Z_{2i}^2\right]=O(1)$. In consequence,

$$
\begin{aligned}
T_2 &= \sum_{k=0}^{\infty}\left\{\frac{e^{-f_q(\gamma_0)v_1/\|\delta_0\|^2}\left(f_q(\gamma_0)\frac{v_1}{\|\delta_0\|^2}\right)^k}{k!}\prod_{i=1}^{k}\left[1+\sqrt{-1}\left(t_1+t_2\right)\frac{\mathbb{E}[z_{2i}]}{\sigma_\delta}-\frac{1}{2}\left(t_1+t_2\right)^2\frac{\mathbb{E}\left[z_{2i}^2\right]}{\sigma_\delta^2}+o\left(\|\delta_0\|^2\right)\right]\right\} \\
&\approx e^{-f_q(\gamma_0)v_1/\|\delta_0\|^2}\sum_{k=0}^{\infty}\frac{\left(f_q(\gamma_0)\frac{v_1}{\|\delta_0\|^2}\right)^k}{k!}\left(1+\sqrt{-1}\left(t_1+t_2\right)\frac{\|\delta_0\|^2Q_\delta}{\sigma_\delta}-\frac{1}{2}\left(t_1+t_2\right)^2\frac{\mathbb{E}\left[z_{2i}^2\right]}{\frac{f_q(\gamma_0)}{\|\delta_0\|^2}\mathbb{E}\left[z_{2i}^2\right]}\right)^k \\
&= e^{-f_q(\gamma_0)v_1/\|\delta_0\|^2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(f_q(\gamma_0)\frac{v_1}{\|\delta_0\|^2}+\sqrt{-1}\left(t_1+t_2\right)\frac{f_q(\gamma_0)Q_\delta v_1}{\sigma_\delta}-\frac{1}{2}\left(t_1+t_2\right)^2v_1\right)^k \\
&= e^{-f_q(\gamma_0)v_1/\|\delta_0\|^2}\exp\left\{f_q(\gamma_0)\frac{v_1}{\|\delta_0\|^2}+\sqrt{-1}\left(t_1+t_2\right)\frac{f_q(\gamma_0)Q_\delta v_1}{\sigma_\delta}-\frac{1}{2}\left(t_1+t_2\right)^2v_1\right\} \\
&= \exp\left\{\sqrt{-1}\left(t_1+t_2\right)\frac{f_q(\gamma_0)Q_\delta v_1}{\sigma_\delta}-\frac{1}{2}\left(t_1+t_2\right)^2v_1\right\}.
\end{aligned}
$$

Similarly,

$$
T_3\approx\exp\left\{\sqrt{-1}t_2\frac{f_q(\gamma_0)Q_\delta\left(v_2-v_1\right)}{\sigma_\delta}-\frac{1}{2}t_2^2\left(v_2-v_1\right)\right\}.
$$

In summary,

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left\{\sqrt{-1}\left[t_1Z_\delta\left(v_1\right)+t_2Z_\delta\left(v_2\right)\right]\right\}\right] \\
&\approx \exp\left\{\sqrt{-1}\left[-t_1\frac{f_q(\gamma_0)Q_\delta v_1}{\sigma_\delta}-t_2\frac{f_q(\gamma_0)Q_\delta v_2}{\sigma_\delta}\right]\right\}\exp\left\{\sqrt{-1}\left(t_1+t_2\right)\frac{f_q(\gamma_0)Q_\delta v_1}{\sigma_\delta}-\frac{1}{2}\left(t_1+t_2\right)^2v_1\right\} \\
&\quad\cdot\exp\left\{\sqrt{-1}t_2\frac{f_q(\gamma_0)Q_\delta(v_2-v_1)}{\sigma_\delta}-\frac{1}{2}t_2^2\left(v_2-v_1\right)\right\} \\
&= \exp\left\{-\frac{1}{2}\left(t_1+t_2\right)^2v_1-\frac{1}{2}t_2^2\left(v_2-v_1\right)\right\} \\
&= \exp\left\{-\frac{1}{2}\left(t_1,t_2\right)\begin{pmatrix} v_1 & v_1 \\ v_1 & v_2\end{pmatrix}\begin{pmatrix} t_1 \\ t_2\end{pmatrix}\right\},
\end{aligned}
$$

giving the required result.

To prove (ii), note that

$$
\begin{aligned}
&P\left(\sup_{|v_2-v_1|<\Delta}\left|Z_\delta\left(v_2\right)-Z_\delta\left(v_1\right)\right|>\epsilon\right)=P\left(\sup_{|v_2-v_1|<\Delta}\left|\frac{\widetilde{D}(v_2)-\widetilde{D}(v_1)}{\sigma_\delta}\right|>\epsilon\right) \\
&\le \mathbb{E}\left[\sup_{|v_2-v_1|<\Delta}\left|\widetilde{D}\left(v_2\right)-\widetilde{D}\left(v_1\right)\right|^2\right]\Big/\sigma_\delta^2\epsilon^2\le 4\mathbb{E}\left[\left|\widetilde{D}\left(v_1+\Delta\right)-\widetilde{D}\left(v_1\right)\right|^2\right]\Big/\sigma_\delta^2\epsilon^2 \\
&= \frac{4\sigma_\delta^2\Delta}{\sigma_\delta^2\epsilon^2}=C\frac{\Delta}{\epsilon^2},
\end{aligned}
$$

where the last inequality is from Doob's martingale inequality with the submartingale defined as $\left|\widetilde{D}\left(s+v_1\right)-\widetilde{D}\left(v_1\right)\right|$ indexed by $0\le s\le\Delta$. So we can choose Δ arbitrarily small to make the above probability smaller than the given η.

Finally, we confirm that $C\left(v\right)\in\mathbf{C}_{\min}\left(\mathbb{R}\right)$. It is not hard to check that $C(v)$ is continuous, has a unique minimum (see Lemma 2.6 of Kim and Pollard (1990)), and $\lim_{|v|\to\infty}C(v)=\infty$ almost surely, which follows since $\lim_{|v|\to\infty}W_\ell\left(v\right)/\left|v\right|=0$ almost surely by virtue of the law of the iterated logarithm for Brownian motion. ■

Additional Reference:

Protter, P.E., 2004, Stochastic Integration and Differential Equations, 2nd ed., New York: Springer.