Chapter 3. Convex Sets and Concave Functions*

Convexity is one of the most important mathematical properties in economics. For example, without convexity of preferences, demand and supply functions are not continuous, and so competitive markets generally do not have equilibrium points. The economic interpretation of convex preference sets in consumer theory is diminishing marginal rates of substitution; the interpretation of convex production sets is constant or decreasing returns to scale. Considerably less is known about general equilibrium models that allow non-convex production sets (e.g., economies of scale) or non-convex preferences (e.g., the consumer prefers a pint of beer or a shot of vodka alone to any mixture of the two). We refer to Nikaido (1968), Rockafellar (1970) and Boyd and Vandenberghe (2004) for more general discussions on convexity; see Hiriart-Urruty and Lemaréchal (2001) for an introduction.

The emphasis of this chapter is to show uniqueness of the optimizer and sufficient conditions for optimization through convexity. Related materials can be found in Chapter 21 of Simon and Blume (1994) and Chapter 7-8 of Sundaram (1996). In this chapter, lowercase bold letters such as $\mathbf{x} = (x_1, \ldots, x_n)'$ represent column vectors.

1 Convex Sets

Given two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, a point $\mathbf{z} = t\mathbf{x} + (1-t)\,\mathbf{y}$, where $0 \leq t \leq 1$, is called a **convex combination** of $\mathbf{x}$ and $\mathbf{y}$. The set of all possible convex combinations of $\mathbf{x}$ and $\mathbf{y}$, denoted by $[\mathbf{x}, \mathbf{y}]$, is called the **interval** with endpoints $\mathbf{x}$ and $\mathbf{y}$ (or, the **line segment** connecting $\mathbf{x}$ and $\mathbf{y}$), i.e.,

$$[\mathbf{x}, \mathbf{y}] = \{t\mathbf{x} + (1-t)\,\mathbf{y} \mid 0 \leq t \leq 1\}.$$

This definition is an extension of the interval in $\mathbb{R}^1$.

Definition 1 *A set $S \subseteq \mathbb{R}^n$ is **convex** iff for any points $\mathbf{x}$ and $\mathbf{y}$ in S the interval $[\mathbf{x}, \mathbf{y}] \subseteq S$.*

In words: a set is convex if it contains the line segment connecting any two of its points; or, more loosely speaking, a set is convex if for any two points in the set it also contains all points between them.

Convex sets in $\mathbb{R}^2$ include triangles, squares, circles, ellipses, and hosts of other sets. Note also that, for example in $\mathbb{R}^3$, while a cube is a convex set, its boundary is not. (Of course, the same is

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true of the square in $\mathbb{R}^2$.) The quintessential convex set in Euclidean space $\mathbb{R}^n$ for any $n > 1$ is the n-dimensional open ball $B_r(\mathbf{a})$ of radius $r > 0$ about point $\mathbf{a} \in \mathbb{R}^n$, where recall from Chapter 1 that

$$B_r(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - a\| < r\}.$$

Example 1 *Prove that the budget constraint $B = \{\mathbf{x} \in X : \mathbf{p}'\mathbf{x} \leq y\}$ is convex.*

Proof. For any two points $\mathbf{x}_1, \mathbf{x}_2 \in B$, we have $\mathbf{p}'\mathbf{x}_1 \leq y$ and $\mathbf{p}'\mathbf{x}_2 \leq y$. Then for any $t \in [0, 1]$, we must have $t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \leq y$. This is equivalent to say that $t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in B$. So the budget constraint B is convex. ■

Exercise 1 *Is the empty set convex? Is a singleton convex? Is $\mathbb{R}^n$ convex? Is the intersection of arbitrarily many convex sets is convex? In each case prove that the set is convex or prove that it is not.*

2 Convex Functions

In order to be able to say whether or not the optimization problem has a unique solution it is useful to know something about the shape or *curvature* of the functions f and $(\mathbf{g}, \mathbf{h})$. Also, the curvature of f and $(\mathbf{g}, \mathbf{h})$ is related to whether the Kuhn-Tucker conditions are sufficient to identify the maximizers.

2.1 Basics

We say a function is *concave* if for any two points in the domain of the function the value of function at a weighted average of the two points is greater than the weighted average of the values of the function at the two points. We say the function is *convex* if the value of the function at the average is less than the average of the values. The following definition makes this a little more explicit.

Definition 2 *A function $f : S \to \mathbb{R}$ defined on a convex set S is **concave** if for any $\mathbf{x}, \mathbf{x}' \in S$ with $\mathbf{x} \neq \mathbf{x}'$ and for any t such that $0 < t < 1$ we have $f(t\mathbf{x} + (1-t)\mathbf{x}') \geq tf(\mathbf{x}) + (1-t)f(\mathbf{x}')$. The function is **strictly concave** if $f(t\mathbf{x} + (1-t)\mathbf{x}') > tf(\mathbf{x}) + (1-t)f(\mathbf{x}')$.*

*A function $f : S \to \mathbb{R}$ defined on a convex set S is **convex** if for any $\mathbf{x}, \mathbf{x}' \in S$ with $\mathbf{x} \neq \mathbf{x}'$ and for any t such that $0 < t < 1$ we have $f(t\mathbf{x} + (1-t)\mathbf{x}') \leq tf(\mathbf{x}) + (1-t)f(\mathbf{x}')$. The function is **strictly convex** if $f(t\mathbf{x} + (1-t)\mathbf{x}') < tf(\mathbf{x}) + (1-t)f(\mathbf{x}')$.*

Exercise 2 *Why don't we check $t = 0$ and 1 in the definition? Why the domain of f must be a convex set?*

Exercise 3 *We say that the function $f(x_1, \cdots, x_n)$ is **nondecreasing** if $x_i' \geq x_i$ for each i implies that $f(x_1', \cdots, x_n') \geq f(x_1, \cdots, x_n)$, is **increasing** if $x_i' > x_i$ for each i implies that $f(x_1', \cdots, x_n') > f(x_1, \cdots, x_n)$ and is **strictly increasing** if $x_i' \geq x_i$ for each i and $x_j' > x_j$ for at least one j implies that $f(x_1', \cdots, x_n') > f(x_1, \cdots, x_n)$. Show that if f is nondecreasing and strictly concave then it must be strictly increasing. [Hint: This is very easy.]*

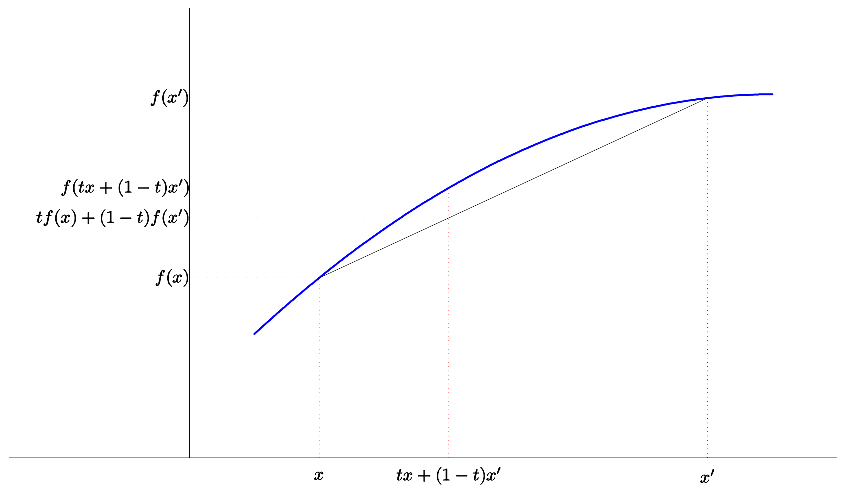

Figure 1: Concave Function

Figures 1 and 2 show a typical concave and convex function intuitively. Note that although there are both concave and convex functions, there are only convex sets, no concave sets!

In practice, we need some calculus criteria for concavity and convexity.

Theorem 1 *Let $f \in C^2(U)$, where $U \subset \mathbb{R}^n$ is open and convex. Then f is concave iff the* ***Hessian***

$$D^2 f(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_n^2} \end{pmatrix}$$

is negative semidefinite for all $\mathbf{x} \in U$. If $D^2 f(\mathbf{x})$ is negative definite for all $\mathbf{x} \in U$, then f is ***strictly*** *concave on U. Conditions for convexity are obtained by replacing "negative" by "positive".*

Remark 1 *The conditions for strict concavity in the theorem are only sufficient, not necessary. In other words, if $D^2 f(\mathbf{x})$ is not negative semidefinite for all $\mathbf{x} \in U$, then f is not concave; while if $D^2 f(\mathbf{x})$ is not negative definite for all $\mathbf{x} \in U$, then f may or may not be strictly concave (see the example below).*

Remark 2 *For a matrix $\mathbf{A}$, we often use $\mathbf{A} > 0$ to denote it is positive definite and $\mathbf{A} \geq 0$ to denote it is positive semidefinite. $\mathbf{A} < 0$ and $\mathbf{A} \leq 0$ can be similarly understood.*

We now define and characterize positive/negative (semi)definite matrices.

Definition 3 *An $n \times n$ matrix $\mathbf{H}$ is* ***positive definite*** *iff $\mathbf{v}'\mathbf{H}\mathbf{v} > 0$ for all $\mathbf{v} \neq \mathbf{0}$ in $\mathbb{R}^n$; $\mathbf{H}$ is* ***negative definite*** *iff $\mathbf{v}'\mathbf{H}\mathbf{v} < 0$ for all $\mathbf{v} \neq \mathbf{0}$ in $\mathbb{R}^n$. Replacing the strict inequalities above by weak ones yields the definitions of* ***positive semidefinite*** *and* ***negative semidefinite****.*

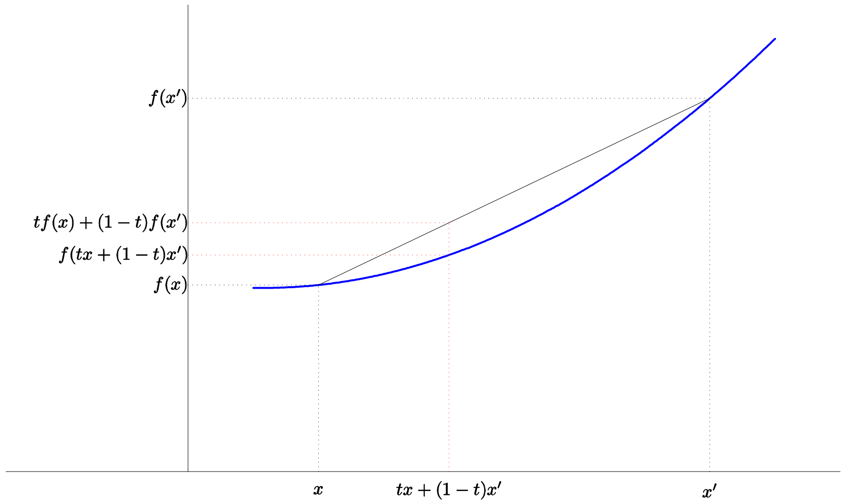

Figure 2: Convex Function

Remark 3 *Usually, positive (negative) definiteness is only defined for a symmetric matrix, so we restrict our discussions on symmetric matrices below. Fortunately, the Hessian is symmetric by Young's theorem.*

Remark 4 *The positive definite matrix is an extension of the positive number. To see why, note that for any positive number H, and any real number $v \neq 0$, $v'Hv = v^2H > 0$. Similarly, the positive semidefinite matrix, negative definite matrix, negative semidefinite matrix are extensions of the nonnegative number, negative number and nonpositive number, respectively.*

Remark 5 *Just like that a function can be neither increasing or decreasing, a matrix can be neither positive (semi)definite or negative (semi)definite, e.g.,* $\mathbf{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *(why? Let* $\mathbf{v} = (1,1)'$ *and* $(-1,1)'$*)*

Exercise 4 *Show that the diagonal elements of a positive definite matrix must be positive, while the off-diagonal elements need not be.*

We now provide some criteria to judge the positive/negative (semi)definiteness. For this purpose, we first define some terms.

Definition 4 *For an $n \times n$ matrix $\mathbf{H}$, a $k \times k$ submatrix formed by picking out k columns and the same k rows is called a kth order* ***principal submatrix*** *of $\mathbf{H}$; the determinant of a kth order principal submatrix is called a kth order* ***principal minor****. The $k \times k$ submatrix formed by picking out the first k columns and the first k rows is called a kth order* ***leading principal submatrix*** *of $\mathbf{H}$; its determinant is called the kth order* ***leading principal minor****.*

Theorem 2 *A matrix is positive definite iff its n* ***leading principal minors*** *are all positive. A matrix is negative definite iff its n leading principal minors alternate in sign with the odd order ones being negative and the even order ones being positive. A matrix is positive semidefinite iff its $2^n - 1$* ***principal minors*** *are all nonnegative. A matrix is negative semidefinite iff its $2^n - 1$ principal minors alternate in sign so that the odd order ones are nonpositive and the even order ones are nonnegative.*

Example 2 *(i) A linear function $f(\mathbf{x}) = a_1x_1 + \cdots + a_nx_n$ is both concave and convex.*

(ii) $f(x) = -x^4$ is strictly concave, but its Hessian is not negative definite for all $x \in \mathbb{R}$ since $D^2f(0) = 0$.

(iii) The particular Cobb-Douglas utility function $u(x_1, x_2) = \sqrt{x_1}\sqrt{x_2}$, $(x_1, x_2) \in \mathbb{R}^2_+$, is concave but not strictly concave. First check that it is concave.

$$D^2f(\mathbf{x}) = \begin{pmatrix} \frac{1}{2}\left(-\frac{1}{2}\right)\frac{\sqrt{x_2}}{\sqrt{x_1^3}} & \frac{1}{2}\frac{1}{2}\frac{1}{\sqrt{x_1}\sqrt{x_2}} \\ \frac{1}{2}\frac{1}{2}\frac{1}{\sqrt{x_1}\sqrt{x_2}} & \frac{1}{2}\left(-\frac{1}{2}\right)\frac{\sqrt{x_1}}{\sqrt{x_2^3}} \end{pmatrix}.$$

Since

$$\frac{1}{2}\left(-\frac{1}{2}\right)\frac{\sqrt{x_2}}{\sqrt{x_1^3}} \le 0, \frac{1}{2}\left(-\frac{1}{2}\right)\frac{\sqrt{x_1}}{\sqrt{x_2^3}} \le 0$$

and

$$\left(\frac{1}{2}\left(-\frac{1}{2}\right)\frac{\sqrt{x_2}}{\sqrt{x_1^3}}\right)\left(\frac{1}{2}\left(-\frac{1}{2}\right)\frac{\sqrt{x_1}}{\sqrt{x_2^3}}\right) - \left(\frac{1}{2}\frac{1}{2}\frac{1}{\sqrt{x_1}\sqrt{x_2}}\right)^2 = 0$$

for $(x_1, x_2) \in \mathbb{R}^2_+$, $u(x_1, x_2)$ is concave. Now, let $x_2 = x_2' = 0$, $x_1 \neq x_1'$; then $u(tx_1 + (1-t)x_1', 0) = 0 = tu(x_1, 0) + (1-t)u(x_1', 0)$, so $u(x_1, x_2)$ is not strictly concave.

2.2 The Uniqueness Theorem

We present two results which indicate the importance of convexity in optimization theory. The first result establishes that in concave optimization problems, all local optima must also be global optima; and, therefore, to find a global optimum in such problems, it always suffices to locate a local optimum. The second result shows that if a strictly concave optimization problem admits a solution, the solution must be unique. We state the two theorem in the context of the mixed constrained maximization problem, i.e.,

$$\begin{aligned} &\max_{\mathbf{x}} \ f(\mathbf{x}) \\ \text{s.t. } &\mathbf{g}(\mathbf{x}) \ge \mathbf{0}, \\ &\mathbf{h}(\mathbf{x}) = \mathbf{0}, \end{aligned}$$

and suppose the maximizer exists. Recall also that the feasible set $G = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{g}(\mathbf{x}) \ge \mathbf{0}, \mathbf{h}(\mathbf{x}) = \mathbf{0}\}$.

Theorem 3 *If f is* ***concave****, and the feasible set G is convex, then*

(i) *Any local maximum of f is a global maximum of f.*

(ii) *The set* $\arg\max\{f(\mathbf{x})|\mathbf{x} \in G\}$ *is convex.*[1]

Theorem 4 *If f is* ***strictly concave****, and the feasible set G is convex, then the maximizer $\mathbf{x}^*$ is unique.*

Proof. Suppose f has two maximizers, say, $\mathbf{x}$ and $\mathbf{x}'$; then $t\mathbf{x}+(1-t)\mathbf{x}' \in G$, and by the definition of strict concavity, for $0 < t < 1$,

$$f(t\mathbf{x}+(1-t)\mathbf{x}') > tf(\mathbf{x})+(1-t)f(\mathbf{x}') = f(\mathbf{x}) = f(\mathbf{x}').$$

A contradiction. ■

Example 3 (Consumer's Problem - Revisited) *Does the consumer's problem*

$$\max_{x_1,x_2} \sqrt{x_1}\sqrt{x_2} \text{ s.t. } x_1+x_2 \leq 1, x_1 \geq 0, x_2 \geq 0$$

have a solution? Is the solution unique?

Solution: The feasible set $G = \{x_1+x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\}$ is compact (why?) and $\sqrt{x_1}\sqrt{x_2}$ is continuous, so by the Weierstrass Theorem, there exists a solution.

The solution is unique, $(x_1^*, x_2^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$. But from the discussion above, $\sqrt{x_1}\sqrt{x_2}$ is not strictly concave for $(x_1, x_2) \in \mathbb{R}^2_+$. Actually, even if we restrict $(x_1, x_2) \in \mathbb{R}^2_{++}$, where $\mathbb{R}_{++} \equiv \{x|x > 0\}$, $\sqrt{x_1}\sqrt{x_2}$ is NOT strictly concave. Check for $t \in (0,1), x_1 \neq x_1'$ and/or $x_2 \neq x_2'$,

$$\begin{aligned}
&\sqrt{tx_1+(1-t)x_1'}\sqrt{tx_2+(1-t)x_2'} \geq t\sqrt{x_1x_2}+(1-t)\sqrt{x_1'x_2'} \\
\iff &(tx_1+(1-t)x_1')(tx_2+(1-t)x_2') \geq \left(t\sqrt{x_1x_2}+(1-t)\sqrt{x_1'x_2'}\right)^2 \\
\iff &x_1x_2'+x_1'x_2 \geq 2\sqrt{x_1x_2x_1'x_2'} \iff \left(\sqrt{x_1x_2'}-\sqrt{x_1'x_2}\right)^2 \geq 0
\end{aligned}$$

with equality holding when $x_2/x_1 = x_2'/x_1'$ (what does this mean?).

In summary, the theorem provides only sufficient (but not necessary) conditions. □

One key question in the uniqueness theorem is how to guarantee that G is convex. Given a concave function g, for any $a \in \mathbb{R}$, its upper contour set $\{\mathbf{x}|g(\mathbf{x}) \geq a\}$ is convex. Why? Given two poitns $\mathbf{x}$ and $\mathbf{x}'$ such that $g(\mathbf{x}) \geq a$ and $g(\mathbf{x}') \geq a$, we want to show that for any $t \in [0,1]$, $g(t\mathbf{x}+(1-t)\mathbf{x}') \geq a$. Since g is concave, $g(t\mathbf{x}+(1-t)\mathbf{x}') \geq tg(\mathbf{x})+(1-t)g(\mathbf{x}') \geq ta+(1-t)a = a$. Given a function h, to guarantee that $\{\mathbf{x}|h(\mathbf{x}) = a\}$ is convex, we require h to be both concave and convex. A function h is both concave and convex iff it is linear (or, more properly, **affine**), taking the form $h(\mathbf{x}) = a+\mathbf{b}'\mathbf{x}$ for some constants a and $\mathbf{b}$. In summary, since

$$G = \left(\bigcap_{j=1}^{J} \{\mathbf{x}|g_j(\mathbf{x}) \geq 0\}\right) \bigcap \left(\bigcap_{k=1}^{K} \{\mathbf{x}|h_k(\mathbf{x}) = 0\}\right),$$

if g_j, $j = 1, \ldots, J$, is concave, and h_k, $k = 1, \cdots, K$, is affine, then G is convex.[2]

[1] Even if $\arg\max\{f(\mathbf{x})|\mathbf{x} \in G\} = \emptyset$, this is still correct since $\emptyset$ is convex from Exercise 1.

[2] From Exercise 1, the intersection of arbitrarily many convex sets is convex.

2.3 Sufficient Conditions for Optimization

We now state the Theorem of Kuhn-Tucker under concavity which especially provides some sufficient conditions for optimization.

Theorem 5 (Theorem of Kuhn-Tucker under Concavity) *Suppose f, g_j and h_k, $j = 1, \cdots, J$, $k = 1, \cdots, K$, are all C^1 function, f is concave, g_j is concave, and h_k is affine. If there exists $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ such that $(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ satisfies the Kuhn-Tucker conditions, then $\mathbf{x}^*$ solves the mixed constrained maximization problem.*

Note that we do not need the NDCQ for this sufficient condition of optimization. Combining with the necessity of Kuhn-Tucker conditions, we can state the necessary and sufficient conditions for optimization which are omitted here to save space.

Example 4 *In Example 3, $g_1(\mathbf{x}) = x_1, g_2(\mathbf{x}) = x_2$ and $g_3(\mathbf{x}) = 1 - x_1 - x_2$ are all affine, so G is convex. Since $u(x_1, x_2) = \sqrt{x_1}\sqrt{x_2}$ is concave, the solution to the Kuhn-Tucker conditions is the global maximizer.*

3 Second Order Conditions for Optimization (*)

As mentioned in Section 1.2 of Chapter 2, the FOCs cannot determine whether their solutions are (local) maximizers or minimizers. The second order conditions (SOCs) do the job. Also, when the set of critical points is large, the second order conditions (SOCs) may provide further refinements. We will provide both necessary and sufficient SOCs. We first consider the equality-constrained problem and then treat the inequality-constrained problem as an extension.

Theorem 6 *In the equality-constrained maximization problem, suppose that $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^K$ are C^2 functions, and the FOCs and NDCQ are satisfied. Define $\mathcal{C}(\mathbf{x}^*) = \{\mathbf{v} \in \mathbb{R}^n | D\mathbf{h}(\mathbf{x}^*)\mathbf{v} = \mathbf{0}_K\}$ as the linear constraint set and let the $n \times n$ matrix $D^2\mathcal{L}^*$ denote the Hessian matrix of $\mathcal{L}$ at $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$, i.e., $D^2\mathcal{L}^* = D^2_{\mathbf{x}}\mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = D^2 f(\mathbf{x}^*) + \sum_{k=1}^K \lambda_k^* \left(D^2 h_k(\mathbf{x}^*)\right)$.*

(i) *If f has a local maximum on G at $\mathbf{x}^*$, then $\mathbf{v}' \left(D^2\mathcal{L}^*\right) \mathbf{v} \leq 0$ for all $\mathbf{v} \in \mathcal{C}(\mathbf{x}^*)$.*

(ii) *If $\mathbf{v}' \left(D^2\mathcal{L}^*\right) \mathbf{v} < 0$ for all $\mathbf{v} \in \mathcal{C}(\mathbf{x}^*)$ with $\mathbf{v} \neq \mathbf{0}$, then $\mathbf{x}^*$ is a strict local maximizer of f on G.*

Exercise 5 *State the SOCs for the equality-constrained minimization problem.*

Exercise 6 *State the SOCs for the unconstrained problem.*

Exercise 7 *Show that $\mathcal{C}(\mathbf{x}^*)$ is the tangent space to $\{\mathbf{h}(\mathbf{x}) = \mathbf{c}\}$ at $\mathbf{x}^*$. Hint: Simon and Blume (1994), pp. 459-460.*

Result (i) is the second order necessary condition and (ii) is the second order sufficient conditions. These conditions are stated in the form of negative definiteness of a symmetric matrix on the linear constraint set. How to check these conditions? The following theorem answer this question.

We first define some notations. For an $n \times n$ symmetric matrix $\mathbf{A}$, $\mathbf{A}_l$ denotes the $l \times l$ submatrix of $\mathbf{A}$ obtained by retaining only the first l rows and columns of $\mathbf{A}$. For a $K \times n$ matrix $\mathbf{B}$, $\mathbf{B}_{Kl}$ denotes the $K \times l$ matrix obtained by retaining only the l columns of $\mathbf{B}$; when $K = l$, denote $\mathbf{B}_{Kl}$ as $\mathbf{B}_K$. Given any permutation π of the first n integers, let $\mathbf{A}^\pi$ denote the $n \times n$ symmetric matrix obtained from $\mathbf{A}$ by applying the permutation π to both its rows and columns and let $\mathbf{B}^\pi$ denote the $K \times n$ matrix obtained by applying the permutation π to only the columns of $\mathbf{B}$. $\mathbf{A}_l^\pi$ and $\mathbf{B}_{Kl}^\pi$ are the $\mathbf{A}_l$ and $\mathbf{B}_{Kl}$ counterpart of $\mathbf{A}^\pi$ and $\mathbf{B}^\pi$. Finally, let $\mathbf{C}_l$ be the $(K + l) \times (K + l)$ matrix obtained by "bordering" the submatrix $\mathbf{A}_l$ by the submatrix $\mathbf{B}_{Kl}$ in the following manner:

$$\mathbf{C}_l = \begin{pmatrix} \mathbf{0}_K & \mathbf{B}_{Kl} \\ \mathbf{B}'_{Kl} & \mathbf{A}_l \end{pmatrix}.$$

Denote by $\mathbf{C}_l^\pi$ the obtained similarly when $\mathbf{A}$ is replaced by $\mathbf{A}^\pi$ and $\mathbf{B}$ by $\mathbf{B}^\pi$. For any matrix $\mathbf{A}$, $|\mathbf{A}|$ denotes $\mathbf{A}$'s determinant.

Theorem 7 *Let $\mathbf{A}$ be a symmetric $n \times n$ matrix, and $\mathbf{B}$ a $K \times n$ matrix such that $|\mathbf{B}_K| \neq 0$. Define the bordered matrices $\mathbf{C}_l$ as described above. Then,*

(i) $\mathbf{x}'\mathbf{A}\mathbf{x} \geq 0$ *for every* $\mathbf{x}$ *such that* $\mathbf{B}\mathbf{x} = \mathbf{0}$ *iff for all permutations π of the first n integers, and for all $r \in \{K + 1, \cdots, n\}$, we have $(-1)^K |\mathbf{C}_r^\pi| \geq 0$.*

(ii) $\mathbf{x}'\mathbf{A}\mathbf{x} \leq 0$ *for every* $\mathbf{x}$ *such that* $\mathbf{B}\mathbf{x} = \mathbf{0}$ *iff for all permutations π of the first n integers, and for all $r \in \{K + 1, \cdots, n\}$, we have $(-1)^r |\mathbf{C}_r^\pi| \geq 0$.*

(iii) $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ *for every* $\mathbf{x}$ *such that* $\mathbf{B}\mathbf{x} = \mathbf{0}$ *iff for all $r \in \{K + 1, \cdots, n\}$, we have $(-1)^K |\mathbf{C}_r| > 0$.*

(ii) $\mathbf{x}'\mathbf{A}\mathbf{x} < 0$ *for every* $\mathbf{x}$ *such that* $\mathbf{B}\mathbf{x} = \mathbf{0}$ *iff for all $r \in \{K + 1, \cdots, n\}$, we have $(-1)^r |\mathbf{C}_r| > 0$.*

When $\mathbf{A} = D^2\mathcal{L}^*$ and $\mathbf{B} = D\mathbf{h}(\mathbf{x}^*)$, the matrices $\mathbf{C}_l$ are called "**bordered Hessians**". We assume $|\mathbf{B}_K| \neq 0$ because rank$(\mathbf{B}) = K$ by the NDCQ and it is without loss of generality to assume the first K columns of $\mathbf{B}$ to be linearly independent.

Exercise 8 *Show that the criteria in the above theorem degenerate to those in Theorem 2 when $K = 0$.*

Exercise 9 *Show that when $\mathbf{A} = D^2\mathcal{L}^*$ and $\mathbf{B} = D\mathbf{h}(\mathbf{x}^*)$, $\mathbf{C}_n = D^2_{\boldsymbol{\lambda},\mathbf{x}}\mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*)$.*

In the inequality-constrained problem, we can replace $\mathbf{h}$ by the union of $\mathbf{h}$ and the binding constraints at $\mathbf{x}^*$ in $\mathbf{g}$. Since for the unbinding constraints the corresponding λ^*'s are zero, the inequality-constrained problem reduces to an equality-constrained problem.

Using the bordered Hessians to check $\mathbf{x}^*$ is a local maximizer or a local minimizer may be burdensome in practice. As an easy (although less general) alternative, we can employ the concavity of the objective function f to draw the conclusion. Specifically, if f is strictly concave at $\mathbf{x}^*$ (or more restrictively, $D^2 f(\mathbf{x}^*) < 0$), then $\mathbf{x}^*$ is a strict local maximizer; if f is strictly convex at $\mathbf{x}^*$ (or more restrictively, $D^2 f(\mathbf{x}^*) > 0$), then $\mathbf{x}^*$ is a strict local minimizer. This is actually the second order sufficient conditions for the unconstrained problem in Exercise 6.