Lecture 07. Confidence Interval Estimation (Sections 7.2-7.8 and Chapter 8)

Ping Yu

HKU Business School
The University of Hong Kong

Plan of This Lecture

- Confidence Interval Estimation: One Population
  - One Normal Mean, Known Population Variance
  - One Normal Mean, Unknown Population Variance
  - One Proportion, Large Samples
  - One Normal Variance
  - Confidence Intervals in Finite Populations
- Sample-Size Determination
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- Confidence Interval Estimation: Two Populations
  - Matched Pair: Two Means
  - Independent Samples: Two Normal Means, Known Population Variances
  - Independent Samples: Two Normal Means, Unknown Equal Population Variances
  - Independent Samples: Two Normal Means, Unknown Unequal Population Variances
  - Independent Samples: Two Proportions, Large Samples
- The discussion of this lecture is parallel to that in the last lecture.

Confidence Interval Estimation: One Population

Confidence Interval (CI)

- A confidence interval estimator of a population parameter is a rule for determining (based on the sample) an interval that is likely to include the parameter. The corresponding estimate is called a confidence interval estimate.
  - This concept was introduced by Jerzy Neyman in 1937.
  - The variability of a point estimator is not reflected in its estimate, but can be reflected in a CI estimate – when the variability is smaller, the CI is typically shorter.
  - The textbook calls a confidence interval estimate as a confidence interval, but we will use "confidence interval" to refer to both "confidence interval estimator" and "confidence interval estimate", depending on the context.

Confidence Level

- Suppose the CI of θ takes the form $[A, B]$, where A and B are random variables, i.e., $[A, B]$ is a random interval. If

$$P(A \leq \theta \leq B) = 1 - \alpha,$$

  then $100(1-\alpha)\%$ is called the confidence level of the CI.
  - We cannot say "θ falls in the CI with $(1-\alpha)$ probability" but only say "the CI covers θ with $(1-\alpha)$ probability", i.e., in repeated samples, $100(1-\alpha)\%$ (realized) intervals will cover θ, where note that given a realization of $[A, B]$, say $[a, b]$, either $\theta \in [a, b]$ or $\theta \notin [a, b]$, but we do not know which happens since θ is unknown.
- Analog: catch a butterfly using a net.

Margin of Error

- Because θ can be larger or smaller than a point estimator $\hat{\theta}$ of θ, the CI typically takes the form

$$\hat{\theta} \pm ME,$$

  where the error factor ME is called the margin of error (or sampling error).
- The width of the CI is equal to twice the ME:

$$w = 2(ME),$$

  which is typically also random.
- The upper confidence limit (UCL) of the CI is given by

$$UCL = \hat{\theta} + ME.$$

- The lower confidence limit (LCL) of the CI is given by

$$UCL = \hat{\theta} - ME.$$

- Either an open interval $(\hat{\theta} - ME, \hat{\theta} + ME)$ or a closed interval $[\hat{\theta} - ME, \hat{\theta} + ME]$ is fine.

One Normal Mean, Known Population Variance

- From the last lecture, for a random sample $\{x_i\}_{i=1}^n$, where $x_i \sim N\left(\mu, \sigma^2\right)$ with unknown μ and known σ^2, if μ_0 is the true value of μ, then

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1),$$

which implies

$$\begin{aligned} 1-\alpha &= P\left(-z_{\alpha/2} \le z \le z_{\alpha/2}\right) = P\left(-z_{\alpha/2} \le \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \le z_{\alpha/2}\right) \\ &= P\left(-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \bar{x} - \mu_0 \le z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \\ &= P\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu_0 \le \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right), \end{aligned}$$

i.e., the interval $\left[\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$ will cover μ_0 with probability $1-\alpha$, so it is a CI with confidence level $100(1-\alpha)\%$ or a $100(1-\alpha)\%$ CI.

- In this example, $\theta = \mu$, $\hat{\theta} = \bar{x}$ and $ME = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.

A General Principle to Construct the CI: Inverting the Test Statistic

- Re-examining the procedure of constructing the CI above, we are actually inverting the test statistic in testing

$$H_0 : \mu = \mu_0 \text{ vs. } H_1 : \mu \neq \mu_0.$$

- Specifically, we try different μ_0's, and for each μ_0 value, we conduct the two-sided test with significance level α; if a μ_0 value is not rejected, then this μ_0 value is put in our CI. The interval collecting all μ_0 values that are not rejected is the CI with confidence level $(1-\alpha)$.
- Conversely, if a value $\mu_0 \notin$CI, then we will reject $H_0 : \mu = \mu_0$ in favor of $H_1 : \mu \neq \mu_0$ at the level of $(1-$confidence level$)$.
- In summary, the hypothesis testing and CI construction are somewhat equivalent.

Table 7.2 Selected Confidence Levels and Corresponding Values of $z_{\alpha/2}$

Confidence Level	90%	95%	98%	99%
α	0.100	0.05	0.02	0.01
$z_{\alpha/2}$	1.645	1.96	2.33	2.58

- The reliability factor $z_{\alpha/2}$ is the critical value for z at the significance level α.

Example 7.3: Time at the Grocery Store

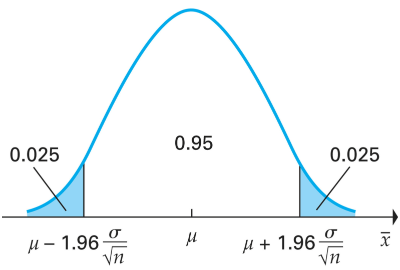
- Suppose the shopping times for customers are normally distributed with population standard deviation 20 minutes. A random sample of 64 shoppers had a mean time of 75 minutes. Construct the 95% CI for the population mean shopping time.
- Solution: Since

$$\bar{x} = 75 \text{ and } \sigma_{\bar{x}} = \sigma/\sqrt{n} = 20/\sqrt{64} = 2.5,$$

we have

$$\begin{aligned} ME &= z_{\alpha/2}\sigma_{\bar{x}} = 1.96 \times 2.5 = 4.9, \\ UCL &= \bar{x} + z_{\alpha/2}\sigma_{\bar{x}} = 75 + 4.9 = 79.9, \\ LCL &= \bar{x} - z_{\alpha/2}\sigma_{\bar{x}} = 75 - 4.9 = 70.1. \end{aligned}$$

So the 95% CI for the population mean shopping time is $[70.1, 79.9]$.



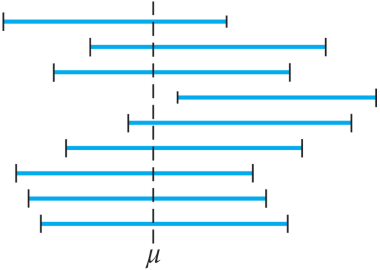


Figure: Sample Distribution of $\bar{x}$ and Schematic Description of 95% CI

- Intuition: $\bar{x}$ more likely appears around μ, so the fixed-length interval $\left[\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$ more likely covers μ.

Reducing Margin of Error

- $ME = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ is decreasing in n and increasing in σ and $(1-\alpha)$.
  - Decreasing in n: if we get more information about the location of the butterfly, then we can use a smaller net.
  - Increasing in σ: if the information about the location of the butterfly is more vague, we must use a larger net.
  - Increasing in $(1-\alpha)$: to catch with a higher probability, we must use a larger net.
- To reduce the width of the CI $(= 2 \cdot ME)$ while maintain the confidence level, we can either increase n or decrease σ (more information or more preciseness).

continue

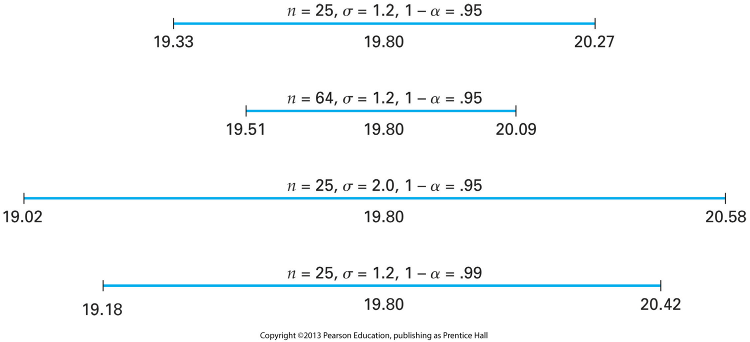


Figure: Effects of n, σ and $(1-\alpha)$ on CIs

One Normal Mean, Unknown Population Variance

- Inverting the test statistic
$$t=\frac{\bar{x}-\mu_0}{s/\sqrt{n}}$$
in testing $H_0:\mu=\mu_0$ vs. $H_1:\mu\neq\mu_0$, we have the $(1-\alpha)$ CI for μ is
$$\left\{\mu_0\middle|\left|\frac{\bar{x}-\mu_0}{s/\sqrt{n}}\right|\leq t_{n-1,\alpha/2}\right\}=\left[\bar{x}-t_{n-1,\alpha/2}\frac{s}{\sqrt{n}},\bar{x}+t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}\right].$$
- $ME=t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}$.
- Compared with $\left[\bar{x}-z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\bar{x}+z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$, this CI should be wider because $t_{n-1,\alpha/2}>z_{\alpha/2}$. This is the cost associated with replacing the unknown σ^2 by s^2.
- When n gets large, $t_{n-1,\alpha/2}\approx z_{\alpha/2}$ and $s\approx\sigma$, so these two CIs are close.
- It is suggested to check whether the data are normally distributed using the normal probability plot whenever the CI construction employs this assumption.

One Proportion, Large Samples

- We can invert the test statistic

$$z=\frac{\hat{p}-p_0}{\sqrt{p_0\left(1-p_0\right)/n}}$$

in testing $H_0: p=p_0$ vs. $H_1: p\neq p_0$, but z is a nonlinear function of p_0, so instead we replace p_0 in the denominator by $\hat{p}$ (which is consistent to p_0 as $n\to\infty$) to have the test statistic

$$\frac{\hat{p}-p_0}{\sqrt{\hat{p}\left(1-\hat{p}\right)/n}}.$$

- The $(1-\alpha)$ CI for p is

$$\left\{p_0 \middle| \left|\frac{\hat{p}-p_0}{\sqrt{\hat{p}\left(1-\hat{p}\right)/n}}\right|\leq z_{\alpha/2}\right\}=\left[\hat{p}-z_{\alpha/2}\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}},\hat{p}+z_{\alpha/2}\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}\right].$$

- $ME=z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.

- The width of the CI can be reduced by either increasing n or decreasing $(1-\alpha)$. [σ is $\sqrt{p\left(1-p\right)}$ here, so out of control]

Example 7.6: Modified Bonus Plan

- Management wants to know the proportion of the corporation's employees who favor a modified bonus plan. From a random sample of 344 employees, it was found that 261 were in favor of the particular plan. What is the 90% CI of the true proportion that favors this plan?
- Solution: First, $\hat{p} = 261/344 = 0.759$. Second, $\alpha = 0.1$, so $z_{\alpha/2} = 1.645$. Therefore, the 90% CI for p is

$$0.759 \pm 1.645\sqrt{\frac{0.759 \times (1 - 0.759)}{344}} = 0.759 \pm 0.038.$$

For the 99% CI for p, the ME increases from 0.038 to

$$2.58\sqrt{\frac{0.759 \times (1 - 0.759)}{344}} = 0.059.$$

One Normal Variance

- Inverting the test statistic

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2},$$

in testing $H_0 : \sigma^2 = \sigma_0^2$ vs. $\sigma^2 \neq \sigma_0^2$, we have the $(1-\alpha)$ CI for σ^2 is

$$\left\{ \sigma_0^2 \left| \chi^2_{n-1,1-\alpha/2} \leq \frac{(n-1)s^2}{\sigma_0^2} \leq \chi^2_{n-1,\alpha/2} \right. \right\} = \left[\frac{(n-1)s^2}{\chi^2_{n-1,\alpha/2}}, \frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/2}} \right].$$

- This CI is not symmetric about s^2 (which is the unbiased estimator of σ^2), so ME is not well defined.
- (*) In general, we can construct the CI for μ and σ^2 based on other tests (i.e., one-sided H_1) in the last lecture. However, such a CI may have an infinite length.

Confidence Intervals in Finite Populations

- n is not much smaller than N in random sampling without replacement, e.g., $n > 0.05N$.
- n itself is large enough so that the CLT can be applied.
- One Mean, Unknown Population Variance, Large Samples: the $(1-\alpha)$ CI for μ is

$$\left[\bar{x} - t_{n-1,\alpha/2}\hat{\sigma}_{\bar{x}}, \bar{x} + t_{n-1,\alpha/2}\hat{\sigma}_{\bar{x}}\right],$$

where

$$\hat{\sigma}_{\bar{x}}^2 = \frac{s^2}{n}\left(\frac{N-n}{N}\right)^1$$

rather than s^2/n is an unbiased estimator of $Var(\bar{x})$.
- Samples need not be drawn from a normal population, but with relatively large n, we can treat the samples in this way.[2]
- $ME = t_{n-1,\alpha/2}\hat{\sigma}_{\bar{x}} < t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}$.

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[1](**) From slide 32 of Lecture 5, $E\left[s^2\right] = S^2 = \frac{N}{N-1}\sigma^2 \neq \sigma^2$ in random sampling without replacement, so the unbiased estimator of $Var(\bar{x})$ $\left(= \frac{\sigma^2}{n}\frac{N-n}{N-1} = \frac{S^2}{n}\frac{N-n}{N}\right)$ should be $\frac{s^2}{n}\frac{N-n}{N}$. Since N is usually large, replacing N by $N-1$ in the textbook does not matter in practice.

[2]In this case, since n is large and $x_i \sim N\left(\mu, \sigma^2\right)$, $t_{n-1,\alpha/2}$ should be changed to $z_{\alpha/2}$.

continue

- The $(1-\alpha)$ CI for the population total $N\mu$ (e.g., the total enrollment in business statistics when μ is the mean enrollment) is

$$\left[N\bar{x}-t_{n-1,\alpha/2}N\hat{\sigma}_{\bar{x}},N\bar{x}+t_{n-1,\alpha/2}N\hat{\sigma}_{\bar{x}}\right].$$

- $ME=t_{n-1,\alpha/2}N\hat{\sigma}_{\bar{x}}$ is N times the ME of the CI for μ.

- One Proportion, Large Samples: the $(1-\alpha)$ CI for p is

$$\left[\hat{p}-z_{\alpha/2}\hat{\sigma}_{\hat{p}},\hat{p}+z_{\alpha/2}\hat{\sigma}_{\hat{p}}\right],$$

where

$$\hat{\sigma}_{\hat{p}}^2=\frac{\hat{p}\left(1-\hat{p}\right)}{n-1}\left(\frac{N-n}{N-1}\right)$$

is an unbiased estimator of $Var\left(\hat{p}\right)$. [see Problem 5(ii) of Assignment III]

- $ME=z_{\alpha/2}\hat{\sigma}_{\hat{p}}=z_{\alpha/2}\sqrt{\frac{(N-n)n}{(N-1)(n-1)}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}<z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ if $n\left(n-1\right)>N-1$ (or roughly $n\succ N^{1/2}$).

Sample-Size Determination

Large Populations

- If we think the CI is too wide, we can narrow it by increasing n.
- Fix the width of the CI, determine how large an n can achieve it.
- Consider only two cases below.
- One Normal Mean, Known Population Variance: Solving

$$ME = z_{\alpha/2}\frac{\sigma}{\sqrt{n}},$$

  we have

$$n = \frac{z_{\alpha/2}^2\sigma^2}{ME^2}, \tag{1}$$

  i.e., to make a $(1-\alpha)$ CI for μ extend a distance ME on each side of $\bar{x}$, we need $\frac{z_{\alpha/2}^2\sigma^2}{ME^2}$ (or $\left\lceil\frac{z_{\alpha/2}^2\sigma^2}{ME^2}\right\rceil$ if $\frac{z_{\alpha/2}^2\sigma^2}{ME^2}$ is not an integer) samples.
- One Proportion, Large Samples: We cannot solve

$$ME = z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},$$

  to have the required n, since $\hat{p}$ is unobserved beforehand.

continue

- Anyway, $\hat{p}(1-\hat{p}) \leq 0.25$, so solving

$$ME = z_{\alpha/2}\sqrt{\frac{0.25}{n}},$$

we conclude that

$$n = \frac{0.25 z_{\alpha/2}^2}{ME^2}$$

can **guarantee** that the CI extends no more than ME on each side of the $\hat{p}$.
- For one normal mean with unknown population variance, $ME = t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}$ is not easy to solve since both $t_{n-1,\alpha/2}$ and s depend on n.

- Note that the ME reported in the media (such as "The poll has a 3% margin of error") includes only the sampling error in $\hat{p}$ and does not include any errors due to biased or otherwise inadequate samples.

- Example 7.14: Electoral College: If an opinion survey on changing the Electoral College process reported that the poll has a 3% margin of error (with 95% confidence), how many citizens of voting age need to be sampled?

- Solution: $n = \frac{0.25 z_{\alpha/2}^2}{ME^2} = \frac{0.25 \times 1.96^2}{0.03^2} = 1067.11 \Longrightarrow n = 1068.$

Finite Populations

- One Mean, Known Population Variance: If $\sigma_{\bar{x}}^2$ is the target, then solving

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right),$$

we have

$$n = \frac{N\sigma^2}{(N-1)\,\sigma_{\bar{x}}^2 + \sigma^2}.$$

- If ME is the target, then solving

$$ME = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$$

to have

$$n = \frac{n_0 N}{n_0 + (N-1)} \le n_0,$$

where n_0 is given in (1). [Implicitly assume normality or the implied n is large]
- If the population total is of interest, then solve $ME = z_{\alpha/2} N \sigma_{\bar{x}}$.

- With nonresponse or missing data, practitioners may add a certain percent (like 10%) to the implied size n.

continue

- One Proportion, Large Samples: Solving

$$\sigma^2_{\hat{p}} = \frac{p(1-p)}{n}\left(\frac{N-n}{N-1}\right)$$

to have

$$n = \frac{Np(1-p)}{(N-1)\sigma^2_{\hat{p}} + p(1-p)}.$$

- Since $p(1-p)$ is unknown, the largest possible value of n is

$$n_{\max} = \frac{0.25N}{(N-1)\sigma^2_{\hat{p}} + 0.25}.$$

- A 95% CI for p will extend approximately $1.96\sigma_{\hat{p}}$ on each side of $\hat{p}$.

- Example 7.16: Campus Survey: Suppose a random sample of the $1,395$ U.S. colleges is taken to estimate the proportion for which the business statistics course is two semesters long. For a 95% CI to extend no further than 0.04 on each side of the sample proportion, how many samples should be taken?
- Solution: $1.96\sigma_{\hat{p}} = 0.04 \Longrightarrow \sigma_{\hat{p}} = 0.020408$. So $n_{\max} = \frac{0.25N}{(N-1)\sigma^2_{\hat{p}}+0.25} = \frac{0.25\times 1{,}395}{1{,}394\times 0.020408^2+0.25} = 419.88 \Longrightarrow n = 420.$

Confidence Interval Estimation: Two Populations

Matched Pair: Two Means

- Inverting the test statistic

$$t = \frac{\bar{d} - \mu_0}{s_d / \sqrt{n}}$$

in testing $H_0 : \mu_d := \mu_x - \mu_y = \mu_0$ vs. $H_1 : \mu_d \neq \mu_0$, we have the $(1 - \alpha)$ CI for μ_d is

$$\left\{ \mu_0 \middle| \left| \frac{\bar{d} - \mu_0}{s_d / \sqrt{n}} \right| \leq t_{n-1,\alpha/2} \right\} = \bar{d} \pm t_{n-1,\alpha/2} \frac{s_d}{\sqrt{n}},$$

where $\bar{d} = \bar{x} - \bar{y}$, and s_d is the sample standard deviation of $\{d_i\}_{i=1}^n$ with $d_i = x_i - y_i$.

- $ME = t_{n-1,\alpha/2} \frac{s_d}{\sqrt{n}}$.

- If the CI contains 0, then we cannot reject $\mu_x = \mu_y$.

Independent Samples: Two Normal Means, Known Population Variances

- Inverting the test statistic

$$z=\frac{(\bar{x}-\bar{y})-\mu_0}{\sqrt{\frac{\sigma_x^2}{n_x}+\frac{\sigma_y^2}{n_y}}}$$

in testing $H_0:\mu_d=\mu_0$ vs. $H_1:\mu_d\neq\mu_0$, we have the $(1-\alpha)$ CI for μ_d is

$$\left\{\mu_0\left|\left|\frac{(\bar{x}-\bar{y})-\mu_0}{\sqrt{\frac{\sigma_x^2}{n_x}+\frac{\sigma_y^2}{n_y}}}\right|\leq z_{\alpha/2}\right.\right\}=(\bar{x}-\bar{y})\pm z_{\alpha/2}\sqrt{\frac{\sigma_x^2}{n_x}+\frac{\sigma_y^2}{n_y}}.$$

- $ME=z_{\alpha/2}\sqrt{\frac{\sigma_x^2}{n_x}+\frac{\sigma_y^2}{n_y}}$.

Independent Samples: Two Normal Means, Unknown Equal Population Variances

- Inverting the test statistic

$$t=\frac{(\bar{x}-\bar{y})-\mu_0}{\sqrt{\frac{s_p^2}{n_x}+\frac{s_p^2}{n_y}}}$$

in testing $H_0:\mu_d=\mu_0$ vs. $H_1:\mu_d\neq\mu_0$, we have the $(1-\alpha)$ CI for μ_d is

$$\left\{\mu_0\left|\left|\frac{(\bar{x}-\bar{y})-\mu_0}{\sqrt{\frac{s_p^2}{n_x}+\frac{s_p^2}{n_y}}}\right|\leq t_{n-2,\alpha/2}\right.\right\}=(\bar{x}-\bar{y})\pm t_{n-2,\alpha/2}\sqrt{\frac{s_p^2}{n_x}+\frac{s_p^2}{n_y}},$$

where $n=n_x+n_y$, and the pooled sample variance

$$s_p^2=\frac{(n_x-1)s_x^2+(n_y-1)s_y^2}{n_x+n_y-2}.$$

- $ME=t_{n-2,\alpha/2}\sqrt{\frac{s_p^2}{n_x}+\frac{s_p^2}{n_y}}$.

Independent Samples: Two Normal Means, Unknown Unequal Population Variances

- Inverting the test statistic

$$t = \frac{(\bar{x} - \bar{y}) - \mu_0}{\sqrt{\frac{s_x^2}{n_x} + \frac{s_y^2}{n_y}}}$$

in testing $H_0 : \mu_d = \mu_0$ vs. $H_1 : \mu_d \neq \mu_0$, we have the $(1 - \alpha)$ CI for μ_d is

$$\left\{ \mu_0 \middle| \left| \frac{(\bar{x} - \bar{y}) - \mu_0}{\sqrt{\frac{s_x^2}{n_x} + \frac{s_y^2}{n_y}}} \right| \leq t_{v,\alpha/2} \right\} = (\bar{x} - \bar{y}) \pm t_{v,\alpha/2} \sqrt{\frac{s_x^2}{n_x} + \frac{s_y^2}{n_y}},$$

where $v = \frac{\left[\left(\frac{s_x^2}{n_x}\right) + \left(\frac{s_y^2}{n_y}\right)\right]^2}{\left(\frac{s_x^2}{n_x}\right)^2 / (n_x - 1) + \left(\frac{s_y^2}{n_y}\right)^2 / (n_y - 1)}$ is defined in the last lecture.

\- $ME = t_{v,\alpha/2} \sqrt{\frac{s_x^2}{n_x} + \frac{s_y^2}{n_y}}$.

\- Whether $\sigma_x^2 = \sigma_y^2$ or not can be tested using the test in the last lecture.

Independent Samples: Two Proportions, Large Samples

- Recall that the test statistic is

$$t=\frac{\hat{p}_x-\hat{p}_y}{\sqrt{\frac{\hat{p}_0(1-\hat{p}_0)}{n_x}+\frac{\hat{p}_0(1-\hat{p}_0)}{n_y}}},$$

  in testing $H_0: p_x-p_y=0$ vs. $H_1: p_x-p_y\neq 0$, which employs the null information $p_x=p_y$ in estimating the variance of $\hat{p}_x-\hat{p}_y$.
- In testing $H_0: p_x-p_y=p_0$ vs. $H_1: p_x-p_y\neq p_0$, the proper test statistic is

$$t=\frac{(\hat{p}_x-\hat{p}_y)-p_0}{\sqrt{\frac{\hat{p}_x(1-\hat{p}_x)}{n_x}+\frac{\hat{p}_y(1-\hat{p}_y)}{n_y}}},$$

  inverting which to have the $(1-\alpha)$ CI for p_x-p_y as

$$\left\{p_0\left|\left|\frac{(\hat{p}_x-\hat{p}_y)-p_0}{\sqrt{\frac{\hat{p}_x(1-\hat{p}_x)}{n_x}+\frac{\hat{p}_y(1-\hat{p}_y)}{n_y}}}\right|\leq z_{\alpha/2}\right.\right\}=(\hat{p}_x-\hat{p}_y)\pm z_{\alpha/2}\sqrt{\frac{\hat{p}_x(1-\hat{p}_x)}{n_x}+\frac{\hat{p}_y(1-\hat{p}_y)}{n_y}}.$$

  - $ME=z_{\alpha/2}\sqrt{\frac{\hat{p}_x(1-\hat{p}_x)}{n_x}+\frac{\hat{p}_y(1-\hat{p}_y)}{n_y}}$.

Comparison of the Neyman-Pearson Approach, Fisher's *p*-Value Approach and Neyman's CI

- Consider testing $H_0 : \mu = \mu_0$ vs. $H_1 : \mu \neq \mu_0$ at the significance level α.
- The Neyman-Pearson approach can only make a decison for a fixed μ_0 and a fixed α each time.
- Fisher's *p*-value approach can make a decision for any α but a fixed μ_0 each time.
- Neyman's CI can make a decision for any μ_0 but a fixed α each time.