Lecture 03. Discrete Random Variables (Chapter 4)

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Plan of This Lecture

- Random Variables
- Probability Distributions for Discrete Random Variables
- Properties of Discrete Random Variables
- Binomial Distribution
- Poisson Distribution
- Hypergeometric Distribution
- Jointly Distributed Discrete Random Variables

Random Variables

Discrete Random Variables

- A random variable (r.v.) is a variable that takes on numerical values realized by the outcomes in the sample space generated by a random experiment.
  - Mathematically, a random variable is a function from S to $\mathbb{R}$.
  - In this and next lectures, we use capital letters, such as X, to denote the random variable, and the corresponding lowercase letter, x, to denote a possible value.
- A discrete random variable is a random variable that can take on no more than a countable number of values.
  - e.g., the number of customers visiting a store in one day, the number of claims on a medical insurance policy in a particular year, etc.
  - "Countable" includes "finite" and "countably infinite".

Continuous Random Variables

- A continuous random variable is a random variable that can take any value in an interval (i.e., for any two values, there is some third value that lies between them).
  - e.g., the yearly income for a family, the highest temperature in one day, etc.
  - The probability can only be assigned to a range of values since the probability of a single value is always zero.
- Recall the distinction between discrete numerical variables and continuous numerical variables in Lecture 1.
- Modeling a r.v. as continuous is usually for convenience as the differences between adjacent discrete values (e.g., \$35,276.21 and \$35,276.22) are of no importance.
- On the other hand, we model a r.v. as discrete when probability statements about the individual possible outcomes have worthwhile meaning.

Probability Distributions for Discrete Random Variables

Probability Distribution Function

- The probability distribution (function), $p(x)$, of a discrete r.v. X represents the probability that X takes the value x, as a function of x, i.e.,

$$p(x) = P(X = x) \text{ for all values of } x.$$

  - Sometimes, the probability distribution of a discrete r.v. is called the probability mass function (pmf).
  - Note that $X = x$ must be an event; otherwise, $P(X = x)$ is not well defined.

- $p(x)$ must satisfy the following properties (implied by the probability postulates in Lecture 2):

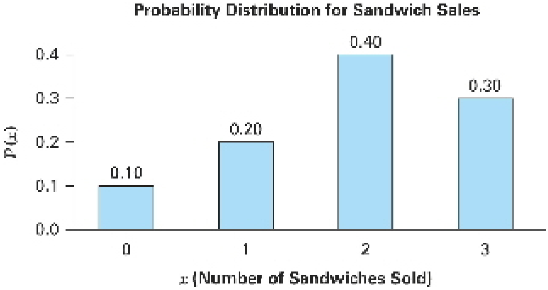
1. $0 \leq p(x) \leq 1$ for any value x,
2. $\sum_{x \in \mathscr{S}} p(x) = 1$, where $\mathscr{S}$ is called the support of X, i.e., the set of all x values such that $p(x) > 0$.

- Notation: I will use $p(x)$ and p (rather than $P(x)$ and P as in the textbook) for pmf and an interested probability to avoid confusion with the probability symbol P.

Example 4.1: Number of Product Sales

Table 4.1 Probability Distribution for Example 4.1

x	$P(x)$
0	0.10
1	0.20
2	0.40
3	0.30



Cumulative Distribution Function

- The cumulative distribution function (cdf), $F(x_0)$, of a r.v. X, represents the probability that X does not exceed the value x_0, as a function of x_0, i.e.,

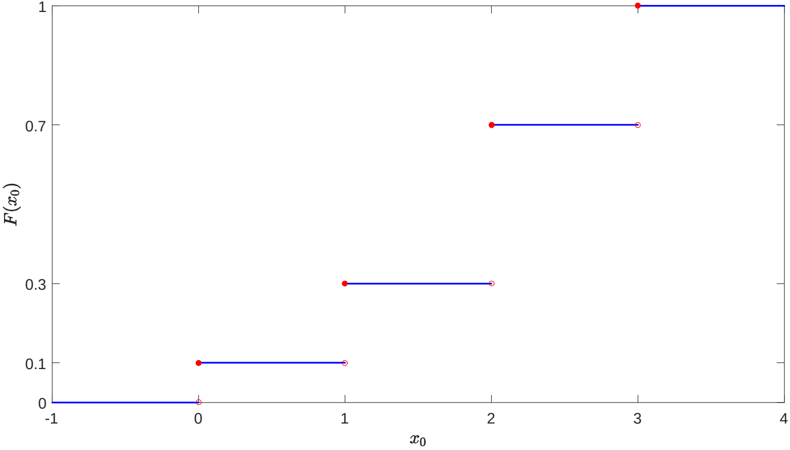
$$F(x_0) = P(X \leq x_0).$$

  - The definition of cdf applies to both discrete and continuous r.v.'s, and $x_0 \in \mathbb{R}$.
  - $F(x_0)$ for a discrete r.v. is a step function with jumps only at support points in $\mathscr{S}$. [figure here]
  - $p(\cdot)$ and $F(\cdot)$ are probabilistic counterparts of histogram and ogive in Lecture 1.
- Relationship between pmf and cdf for discrete r.v.'s:

$$F(x_0) = \sum_{x \leq x_0} p(x).$$

- From the definition of cdf, we have (i) $0 \leq F(x_0) \leq 1$ for any x_0; (ii) if $x_0 < x_1$, $F(x_0) \leq F(x_1)$, i.e., $F(\cdot)$ is an (weakly) increasing function.
- From the figure in the next slide, we can also see (iii) $F(x_0)$ is right continuous, i.e., $\lim_{x \downarrow x_0} F(x) = F(x_0)$; (iv) $\lim_{x_0 \to -\infty} F(x_0) = 0$ and $\lim_{x_0 \to \infty} F(x_0) = 1$.

Example Continued



Properties of Discrete Random Variables

Mean

- The pmf contains all information about the probability properties of a discrete r.v., but it is desirable to have some summary measures of the pmf's characteristics.
- The mean (or expected value, or expectation), $E[X]$, of a discrete r.v. X is defined as

$$E[X] = \mu = \sum_{x \in \mathcal{S}} xp(x).$$

- The mean of X is the same as the population mean in Lecture 1, $\mu = \frac{\sum_{i=1}^{N} x_i}{N}$, but we use the probability language here: think of $E[X]$ in terms of relative frequencies,

$$\frac{\sum_{i=1}^{N} x_i}{N} = \sum_{x \in \mathcal{S}} x \frac{N_x}{N},$$

weighting each possible value x by its probability.
- In other words, the mean of X is is a weighted average of all possible values of X.
- For example, if we roll a die once, the expected outcome is

$$E[X] = \sum_{i=1}^{6} i \times \frac{1}{6} = 3.5.$$

Variance

- The variance, $Var(X)$, of a discrete r.v. X is defined as

$$Var(X) = \sigma^2 = E\left[(X-\mu)^2\right] = \sum_{x\in\mathscr{S}} (x-\mu)^2 p(x).$$

  - This definition of $Var(X)$ is the same as the population variance in Lecture 1.
  - It is not hard to see that

$$\begin{aligned}\sigma^2 &= \sum_{x\in\mathscr{S}} (x-\mu)^2 p(x) = \sum_{x\in\mathscr{S}} x^2 p(x) - 2\mu \sum_{x\in\mathscr{S}} xp(x) + \mu^2 \sum_{x\in\mathscr{S}} p(x) \\ &= E\left[X^2\right] - 2\mu E[X] + \mu^2 = E\left[X^2\right] - 2\mu^2 + \mu^2 \\ &= E\left[X^2\right] - \mu^2,\end{aligned}$$

  i.e., the second moment − first moment^2,[1] where in the third equality, $p(x)$ is the probability of $X^2 = x^2$,[2] and $\sum_{x\in\mathscr{S}} p(x) = 1$.

- The standard deviation, $\sigma = \sqrt{\sigma^2}$, is the same as the population standard deviation in Lecture 1.

[1] σ^2 is also called the second central moment.

[2] What will happen if X can take both 1 and −1? $\sum_{x^2=1} x^2 \times P(X^2 = x^2) = \sum_{x^2=1} x^2 \times (p(1) + p(-1)) = 1^2 \times p(1) + (-1)^2 \times p(-1)$.

Mean of Functions of a R.V.

- For a function of X, $g(X)$, its mean, $E[g(X)]$, is defined as

$$E[g(X)] = \sum_{x \in \mathscr{S}} g(x) p(x).$$

  - e.g., X is the time to complete a contract, and $g(X)$ is the cost when the completion time is X; we want to know the expected cost.

- $E[g(X)] \neq g(E[X])$ in general, e.g., if $g(X) = X^2$, then

$$E[g(X)] - g(E[X]) = E\left[X^2\right] - \mu^2 = \sigma^2 > 0.$$

  - However, when $g(X)$ is linear in X, $E[g(X)] = g(E[X])$.

Mean and Variance of Linear Functions

- For $Y = a + bX$ with a and b being constant fixed numbers,

$$\mu_Y := E[Y] = E[a + bX] = a + bE[X] =: a + b\mu_X,$$
$$\sigma_Y^2 := Var(Y) = Var(a + bX) = b^2 Var(X) =: b^2\sigma_X^2,$$

  and

$$\sigma_Y = \sqrt{Var(Y)} = |b|\sigma_X.$$

  - The proof can follow similar steps as in the last last slide. [Exercise]
  - The constant a will not contribute to the variance of Y.

- Some Special Linear Functions:
  - If $b = 0$, i.e., $Y = a$, then $E[a] = a$ and $Var(a) = 0$.
  - If $a = 0$, i.e., $Y = bX$, then $E[bX] = bE[X]$ and $Var(bX) = b^2 Var(X)$.
  - If $a = -\mu_X/\sigma_X$ and $b = 1/\sigma_X$, i.e., $Y = \frac{X-\mu_X}{\sigma_X}$ is the z-score of X, then

$$E\left[\frac{X-\mu_X}{\sigma_X}\right] = \frac{\mu_X}{\sigma_X} - \frac{\mu_X}{\sigma_X} = 0$$

  and

$$Var\left(\frac{X-\mu_X}{\sigma_X}\right) = \frac{Var(X)}{\sigma_X^2} = 1.$$

Binomial Distribution

Bernoulli Distribution

- The Bernoulli r.v. is a r.v. taking only two values, 0 and 1, labeled as "failure" and "success". [figure here]
- If the probability of success, $p(1) = p$, then the probability of failure, $p(0) = 1 - p(1) = 1 - p$. This distribution is known as the Bernoulli distribution, and we denote a r.v. X with this distribution as $X \sim$Bernoulli(p).
- The mean of a Bernoulli(p) r.v. X is

$$\mu_X = E[X] = 1 \times p + 0 \times (1 - p) = p,$$

and the variance is

$$\begin{aligned} \sigma_X^2 &= Var(X) = (1-p)^2 \times p + (0-p)^2 \times (1-p) \\ &= p(1-p). \end{aligned}$$

- When $p = 0.5$, σ_X^2 achieves its maximum; when $p = 0$ and 1, $\sigma_X^2 = 0$. [why?]

History of the Bernoulli Distribution

Jacob Bernoulli (1655-1705), Swiss

- Jacob Bernoulli (1655-1705) was one of the many prominent mathematicians in the Bernoulli family.

Binomial Distribution

- The binomial r.v. X is the number of successes in n independent trials of a Bernoulli(p) r.v., denoted as $X \sim$Binomial(n,p).
- Denote X_i as the outcome in the ith trial, then the binomial r.v. $X = \sum_{i=1}^{n} X_i$.
- After some thinking, we can figure out that the number of sequences with x successes in n trials is C_x^n, and the probability of any sequence with x successes is $p^x(1-p)^{n-x}$ by the multiplication rule.
- By the addition rule, the binomial distribution is

$$p(x|n,p) = C_x^n p^x (1-p)^{n-x}, x = 0, 1, \cdots, n.$$

- From the discussion on multivariate r.v.'s below, we can show

$$\mu_X = E[X] \overset{(*)}{=} \sum_{i=1}^{n} E[X_i] = np,$$

and

$$\sigma_X^2 = Var(X) \overset{(**)}{=} \sum_{i=1}^{n} Var(X_i) = np(1-p).$$

- (*) holds even if X_i's are dependent, while (**) depends on the independence of X_i's; see the slides on jointly distributed r.v.'s.

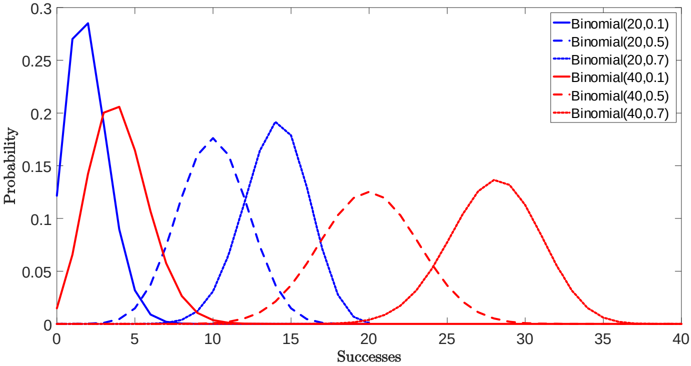


Figure: Binomial Distributions with Different *n* and *p*

Poisson Distribution

Poisson Distribution

- The Poisson distribution was proposed by Siméon Poisson in 1837. [figure here]
- Assume that an interval is divided into a very large number of equal subintervals so that the probability of the occurrence of an event in any subinterval is very small (e.g., ≤ 0.05). The Poisson distribution models the number of events occuring on that inverval, assuming

1. The probability of the occurrence of an event is constant for all subintervals.
2. There can be no more than one occurrence in each subinterval.
3. Occurrences are independent.

- From these assumptions, we can see the Poisson distribution can be used to model, e.g., the number of failures in a large computer system during a given day, the number of ships arriving at a dock during a 6-hour loading period, the number of defective products in large production runs, etc.
- The Poisson distribution is particularly useful in waiting line, or queuing, problems, e.g., the probability of various numbers of customers waiting for a phone line or waiting to check out of large retail store.
  - For a store manager, how to balance long lines (too few checkout lines, losing customers) and idle customer service associates (too many lines, resulting waste)?

History of the Poisson Distribution

Siméon D. Poisson (1781-1840), French

continue

- Intuitively, the Poisson r.v. is the binomial r.v. taking limit as $p \to 0$ and $n \to \infty$. If $np \to \lambda$ which specifies the average number of occurrences (successes) for a particular time (and/or space), then the binomial distribution converges to the Poisson distribution:

$$p(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \cdots,$$

  where $e = 2.71828\cdots$ is the base for natural logarithms, called Euler's number. [proof not required]
  - We denote a r.v. X with the above Poisson distribution as $X \sim$Poisson(λ).
  - When n is large and np is of only moderate size (preferably $np \le 7$), the binomial distribution can be approximated by Poisson(np). [figure here]
- $\mu_X = E[X] = \lambda$, and $\sigma_X^2 = Var(X) = \lambda$.
  - $np \to \lambda$, and $np(1-p) = np - np \cdot p \to \lambda - \lambda \cdot 0 = \lambda$.
- The sum of independent Poisson r.v.'s is also a Poisson r.v., e.g., the sum of K Poisson(λ) r.v. is a Poisson$(K\lambda)$ r.v..

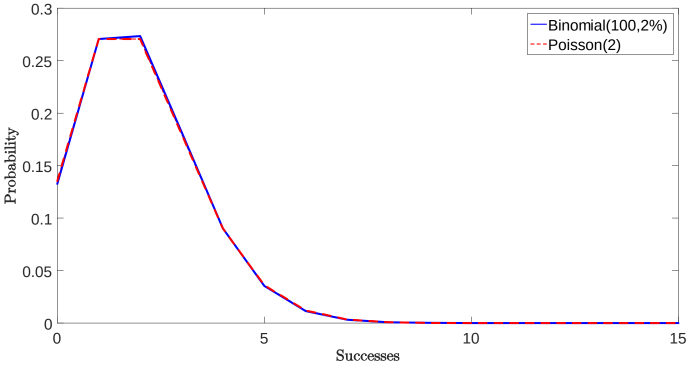


Figure: Poisson Approximation

- For an example whether the approximation is not this good, see Assignment II.8(ii).

Hypergeometric Distribution

Hypergeometric Distribution

- If the binomial distribution can be treated as from random sampling with replacement from a population of size N, S of which are successes and $S/N = p$, then the hypergeometric distribution models the number of successes from random sampling without replacement.
  - These two random sampling schemes will be discussed more in Lecture 5.
- The hypergeometric distribution is

$$p(x|n,N,S) = \frac{C_x^S C_{n-x}^{N-S}}{C_n^N}, x = \max(0, n-(N-S)), \cdots, \min(n,S),$$

  where n is the size of the random sample, and x is number of successes.
  - A r.v. with this distribution is denoted as $X \sim$Hypergeometric(n,N,S).
- The binomial distribution assumes the items are drawn independently, with the probability of selecting an item being constant.
- This assumption can be met in practice if a small sample is drawn (without replacement) from a large population (e.g., $N > 10,000$ and $n/N < 1\%$). [figure here]
- When we draw from a small population, the probability of selecting an item is changing with each selection because the number of remaining items is changing.

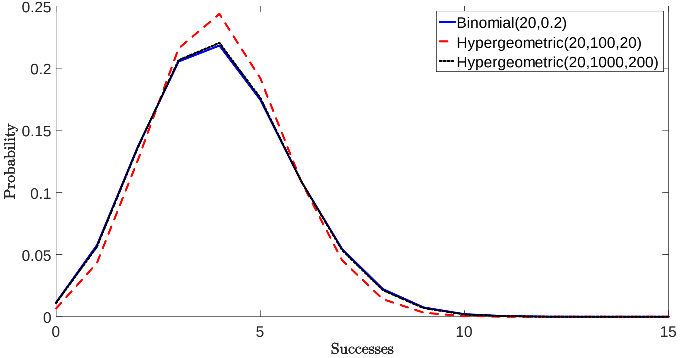


Figure: Comparison of Binomial and Hypergeometric Distributions

- $\mu_X = E[X] = np$, and $\sigma_X^2 = Var(X) = np(1-p)\frac{N-n}{N-1} \leq np(1-p)$,[3] where $p = \frac{S}{N}$. [proof not required]

[3] When $\frac{n}{N}$ is small, $\frac{N-n}{N-1}$ is close to 1, matching the variance of the binomial r.v.

Jointly Distributed Discrete Random Variables

Bivariate Discrete R.V.'s: Joint and Marginal Probability Distributions

- We can use bivariate probability distribution to model the relationship between two univariate r.v.'s.
- For two discrete r.v.'s X and Y, their joint probability distribution expresses the probability that simultaneously X takes the specific value x and Y takes the value y, as a function of x and y:

$$p(x,y) = P(X = x \cap Y = y), x \in \mathscr{S}_X \text{ and } y \in \mathscr{S}_Y.$$

  - $p(x,y)$ is a straightforward extension of joint probabilities in Lecture 2, where $X = x$ and $Y = y$ are two events with x and y indexing them.
  - From probability postulates in Lecture 2, $0 \le p(x,y) \le 1$, and $\sum_{x \in \mathscr{S}_X} \sum_{y \in \mathscr{S}_Y} p(x,y) = 1$.
- The marginal probability distribution of X is

$$p(x) = \sum_{y \in \mathscr{S}_Y} p(x,y),$$

and the marginal probability distribution of Y is

$$p(y) = \sum_{x \in \mathscr{S}_X} p(x,y),$$

Conditional Probability Distribution and Independence of Bivariate R.V.'s

- These two concepts are parallel to conditional probabilities and independent events in Lecture 2.
- The conditional probability distribution of Y, given that X takes the value x, expresses the probability that Y takes the value y, as a function of y, when the value x is fixed for X:
$$p(y|x) = \frac{p(x,y)}{p(x)};$$
similarly, the conditional probability distribution of X, given $Y = y$, is
$$p(x|y) = \frac{p(x,y)}{p(y)}.$$
- One way of thinking of conditioning is filtering a data set based on the value of X.
- Two r.v.'s X and Y are independent iff
$$p(x,y) = p(x)p(y)$$
for all $x \in \mathscr{S}_X$ and $y \in \mathscr{S}_Y$, i.e., independence of r.v.'s can be understood as a set of independencies of events. E.g., "height" and "musical talent" are independent.
- Generally, k r.v.'s are independent if $p(x_1, \cdots, x_k) = p(x_1)p(x_2)\cdots p(x_k)$.
- X and Y are independent iff $p(y|x) = p(y)$ or $p(x|y) = p(x)$ ($\Longrightarrow$symmetric).

Conditional Mean and Variance

- The conditional mean of Y, given that X takes the value x, is given by

$$\mu_{Y|X=x} = E[Y|X=x] = \sum_{y \in \mathscr{S}_Y} y p(y|x).$$

  - For any constants a and b, $E[a+bY|X=x] = a + bE[Y|X=x]$.

- The conditional variance of Y, given that X takes the value x, is given by

$$\sigma^2_{Y|X=x} = Var(Y|X=x) = \sum_{y \in \mathscr{S}_Y} \left(y - \mu_{Y|X=x}\right)^2 p(y|x).$$

  - For any constants a and b, $Var(a+bY|X=x) = b^2 Var(Y|X=x)$.

- Notation: The notations used in the textbook, $\mu_{Y|X}$ and $\sigma^2_{Y|X}$, are not clear.

Mean and Variance of (Linear) Functions

- For a function of (X,Y), $g(X,Y)$, its mean, $E[g(X,Y)]$, is defined as

$$E[g(X,Y)] = \sum_{x\in\mathscr{S}_X}\sum_{y\in\mathscr{S}_Y} g(x,y)p(x,y).$$

- For a linear function of (X,Y), $W = aX + bY$,

$$\mu_W := E[W] = a\mu_X + b\mu_Y, \text{[verified in the next slide]}$$
$$\sigma_W^2 := Var(W) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY} \text{ [see the next}^3 \text{ slide for } \sigma_{XY}\text{].}$$

- e.g., W is the total revenue of two products with (X,Y) being the sales and (a,b) the prices.
- If $a = b = 1$, then $E[X+Y] = E[X] + E[Y]$, i.e., the mean of sum is the sum of means.
- If $a = 1$ and $b = -1$, then $E[X-Y] = E[X] - E[Y]$, i.e., the mean of difference is the difference of means.
- If $a = b = 1$ and $\sigma_{XY} = 0$, then $Var(X+Y) = Var(X) + Var(Y)$, i.e., the variance of sum is the sum of variances.
- If $a = 1, b = -1$ and $\sigma_{XY} = 0$, then $Var(X-Y) = Var(X) + Var(Y)$, i.e., the variance of difference is the <u>sum</u> of variances.

(*) Verification and Extensions

- μ_W:

$$\begin{aligned}
\mu_W &= \sum_{x\in\mathscr{S}_X}\sum_{y\in\mathscr{S}_Y}(ax+by)\,p(x,y) \\
&= a\sum_{x\in\mathscr{S}_X}\left[x\sum_{y\in\mathscr{S}_Y}p(x,y)\right]+b\sum_{y\in\mathscr{S}_Y}\left[y\sum_{x\in\mathscr{S}_X}p(x,y)\right] \\
&= a\sum_{x\in\mathscr{S}_X}xp(x)+b\sum_{y\in\mathscr{S}_Y}yp(y) \\
&= a\mu_X+b\mu_Y,
\end{aligned}$$

- σ_W^2 can be derived based on this result. [Exercise]
- Extension I: If $W=\sum_{i=1}^K a_iX_i$, then

$$\mu_W=E[W]=\sum_{i=1}^K a_iE[X_i]=:\sum_{i=1}^K a_i\mu_i,$$

continue

- and

$$\sigma_W^2 = Var(W) = \sum_{i=1}^K a_i^2 Var(X_i) + 2\sum_{i=1}^{K-1}\sum_{j>i}^K a_i a_j Cov(X_i, X_j)$$
$$=: \sum_{i=1}^K a_i^2 \sigma_i^2 + 2\sum_{i=1}^{K-1}\sum_{j>i}^K a_i a_j \sigma_{ij}.$$

- If $a_i = 1$ for all i, then we have

$$E\left[\sum_{i=1}^K X_i\right] = \sum_{i=1}^K \mu_i \text{ and } Var\left(\sum_{i=1}^K X_i\right) = \sum_{i=1}^K \sigma_i^2 + 2\sum_{i=1}^{K-1}\sum_{j>i}^K \sigma_{ij},$$

where $Var\left(\sum_{i=1}^K X_i\right)$ reduces to $\sum_{i=1}^K \sigma_i^2$ if $\sigma_{ij} = 0$ for all $i \neq j$.

- Extension II: For $W = aX + bY$ and a r.v. Z different from (X, Y),

$$\begin{aligned} E[W|Z=z] &= a\mu_{X|Z=z} + b\mu_{Y|Z=z}, \\ Var(W|Z=z) &= a^2\sigma_{X|Z=z}^2 + b^2\sigma_{Y|Z=z}^2 + 2ab\sigma_{XY|Z=z}, \end{aligned}$$

where $Var(W|Z=z)$ reduces to $a^2\sigma_{X|Z=z}^2 + b^2\sigma_{Y|Z=z}^2$ if $\sigma_{XY|Z=z} = 0$.

Covariance and Correlation

- These two concepts are the same as those in Lecture 1 but in the probability language.
- The covariance between X and Y

$$Cov(X,Y) = \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \sum_{x\in\mathscr{S}_X}\sum_{y\in\mathscr{S}_Y}(x-\mu_X)(y-\mu_Y)p(x,y).$$

  - It is not hard to show that $Cov(X,Y) = E[XY] - \mu_X\mu_Y$, which reduces to $Var(X) = E\left[X^2\right] - \mu_X^2$ when $X = Y$.
- The correlation between X and Y

$$Corr(X,Y) = \rho_{XY} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}.$$

- Recall that σ_{XY} is not unit-free so is unbounded, while $\rho_{XY} \in [-1,1]$ is more useful.
- Recall that σ_{XY} and ρ_{XY} have the same sign: if they are positive, X and Y are called positively dependent, when they are negative, X and Y are called negatively dependent, when they are zero, there is no linear relationship between X and Y.

Covariance and Independence

- If X and Y are independent, then $Cov(X,Y) = Corr(X,Y) = 0$. [Exercise]
  - The converse is not true; recall the figure in Lecture 1.
- Here is a concrete example: if the distribution of X is

$$p(-1) = 1/4, p(0) = 1/2 \text{ and } p(1) = 1/4,$$

  then $Cov(X,Y) = 0$ with $Y = X^2$. Why?
- Because X can determine Y, X and Y are not independent.
- The distribution of X implies $E[X] = 0$.
- The distribution of Y is $p(0) = p(1) = 1/2$, i.e., Y is a Bernoulli r.v., which implies $E[Y] = 1/2$.
- The joint distribution of (X,Y) is

$$p(-1,1) = 1/4, p(0,0) = 1/2, p(1,1) = 1/4,$$

  which implies $E[XY] = 0$, so $Cov(X,Y) = E[XY] - E[X]E[Y] = 0$.
- Portfolio analysis in the textbook (Pages 190-192, 236-240) will be discussed in the next tutorial class.